TOPICS IN ERGODIC THEORY AND RAMSEY THEORY

DISSERTATION

Presented in partial fulfillment of the requirements for the degree Doctor of Philosophy
in the Graduate School of The Ohio State University

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Graduate Program in Mathematics

The Ohio State University
2022

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ABSTRACT

This thesis is comprised of two main parts. The first part consists of topics in ergodic theory. In particular, we deal with variations of van der Corput's difference theorem, van der Corput sets, strengthenings of Birkhoff's ergodic theorem, and a generalization of the notion of uniform distribution. We show that van der Corput's difference theorem in Hilbert spaces and in uniform distribution theory is connected to the ergodic hierarchy of mixing properties. We show that our strengthening of Birkhoff's ergodic theorem leads to a variety of weighted ergodic theorems. We generalize the notion of uniform distribution to that of *uniform symmetric distribution*, and obtain applications to measure preserving systems. We add to the known list of equivalent formulations of van der Corput sets, and answer some open questions from the literature.

In the second part, which is based on joint work with Richard Magner, we give an almost complete classification of those $m, n \in \mathbb{N}$ and $a, b, c \in \mathbb{Z} \setminus \{0\}$ for which the equation $ax + by = cw^m z^n$ is partition regular over $\mathbb{Z} \setminus \{0\}$. This generalizes the result of Bergelson and Hindman about the partition regularity of the equation $x + y = wz$. One of the key ingredients in the proof of our result is a partial generalization of the Grunwald-Wang Theorem. We also prove results of independent interest about ultrafilters q over an infinite integral domain R for which each $A \in q$ has substantial additive and multiplicative structure.

ACKNOWLEDGEMENTS

I would not have gotten far enough in math to write a thesis without the help and support of many others. I would like to thank my advisor Vitaly Bergelson for being generous with his time and ideas, Bhawesh Mishra for helpful mathematical discussions, my middle school math club teachers Libby Hart and Clarence Lynn Fairbrother for getting me interested in math through the Mathcounts competitions, my 10th grade algebra teacher Mary Ann Donahue for believing in my mathematical abilities at a time when I needed it, Dr. Robert Sachs for running the Fairfax Math Circle and supporting me throughout high school, Jonathan Osborne for his enthusiasm teaching high schoolers advanced mathematics, and my masters thesis advisor Ezra Brown for his guidance and support during my time at Virginia Tech. While there are many others who have played a hand in my mathematical journey, I would like to conclude by thanking my cat Gauss Jr. for always reminding me to take a step back and calm down when I am overwhelmed.

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- S. Farhangi. and J. Grytczuk "Distance Graphs and Arithmetic Progressions". In: *Integers*, 21A (Ron Graham Memorial Volume): Paper No. A11, 6, 2021.

FIELDS OF STUDY

Major Field: Mathematics
Specialization: Combinatorics, Ergodic Theory, Uniform Distribution

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CHAPTER 1

INTRODUCTION

1.1 Background

Our story begins in 1916 with a theorem of Issai Schur that was one of the first results in the field now known as Ramsey theory. We use $\mathbb{N}$ to denote the set of positive integers.

Theorem 1.1.1 (cf. [Sch16]). *For any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exists some $1 \le i \le r$ and $x, y \in \mathbb{N}$ for which $x, y, x+y \in C_i$.*

Schur had proven this theorem in order to show that the equation $x^m + y^m = z^m \pmod{p}$ is solvable for any prime p. 11 years later Bartel L. van der Waerden proved a theorem similar in spirit to that of Schur's involving arithmetic progressions.

Theorem 1.1.2 (cf. [Wae27]). *For any $\ell \in \mathbb{N}$ and any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exists some i and $a, d \in \mathbb{N}$ for which $\{a + id\}_{i=0}^{\ell} \subseteq C_i$.*

These theorems naturally lead us to the notion of partition regularity. Letting $\mathcal{P}(\mathbb{N})$ denote the power set of $\mathbb{N}$, a collection $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$ is **partition regular (p.r.)** if for any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exist some $1 \le i \le r$ and some $F \in \mathcal{F}$ with $F \subseteq C_i$. In these terms we can see Schur's theorem as the statement that the collection $\mathcal{F} := \{\{x, y, x+y\} \mid x, y \in \mathbb{N}\}$ is p.r. Similarly, we can see van der Waerden's theorem as the statement that for any $\ell \in \mathbb{N}$ the collection $\mathcal{F}_\ell := \{\{a + id\}_{i=0}^{\ell} \mid a, d \in \mathbb{N}\}$ is p.r. One of the active areas of research in the field of Ramsey theory consists of finding p.r. collections of sets. In 1928 Brauer proved a common refinement of Schur's Theorem and van der Waerden's Theorem by showing that for any $\ell \in \mathbb{N}$ the collection $\mathcal{F}_\ell := \{\{d\} \cup \{a + id\}_{i=0}^{\ell} \mid a, d \in \mathbb{N}\}$ is partition regular (cf. [Bra28]). In 1933 Richard Rado classified those finite systems of linear equations $\mathcal{S}$ for which the collection $\mathcal{S}$ is p.r. (cf. [Rad33]). An exposition of all of these results as well as how to obtain the first three results as a consequence of Rado's theorem can be found in [GRS13]. While Rado's theorem can intuitively be seen as classifying all p.r. families consisting of finite sets with linear structure, the situation for polynomial structures remains wide open. This is discussed in more detail in Chapter 6.1. The purpose of Chapter 6 is to contribute to the understanding of p.r.

families of sets with polynomial structure by providing an almost complete classification of those $m, n \in \mathbb{N}$ and $a, b, c \in \mathbb{Z} \setminus \{0\}$ for which the collection $\mathcal{F}(m, n, a, b, c)$ of solutions to the poynomial equation $ax + by = cw^m z^n$ is partition regular (see Theorem 6.1.1).

Our story takes a turn in 1936 with Conjecture 1.1.4 of Pál Erdős and Pál Turán. To continue the discussion we first require some terminology.

Definition 1.1.3. *Let $A \subset \mathbb{N}$. The* ***upper density of*** A *is given by*

$$\overline{d}(A) = \limsup_{N \to \infty} \frac{1}{N} |A \cap [1, N]|. \tag{1.1}$$

Conjecture 1.1.4 (cf. [ET36])**.** *$A \subset \mathbb{N}$ contains arbitrarily long arithmetic progressions if* $\overline{d}(A) > 0$.

Conjecture 1.1.4 is a *density analogue* of van der Waerden's theorem. While density results always imply the analogous statements in the context of partition regularity, the converse is not always true. To see this, we observe that the odd integers have a natural density of $\frac{1}{2}$ but contain no set of the form $\{x, y, x + y\}$, so the density analogue of Schur's theorem does not hold.[1] It wasn't until 1952 that Klaus Roth answered Conjecture 1.1.4 in the specialized case of 3-term arithmetic progressions.

Theorem 1.1.5 ([Rot52])**.** *$A \subset \mathbb{N}$ contains infinitely many 3-term arithmetic progressions if $\overline{d}(A) > 0$.*

A year later, in [Rot53], Roth further refined Theorem 1.1.5 for even sparser sets A. Progress on Conjecture 1.1.4 laid stagnant for another 16 years before Endre Szemerédi was able to extend Theorem 1.1.5 to the case of 4-term arithmetic progressions in [Sze69]. It then took another 8 years before Szemerédi answered Conjecture 1.1.4 in the positive.

Theorem 1.1.6 (cf. Theorem 1.4 in [Sze75])**.** *$A \subset \mathbb{N}$ contains arbitrarily long arithmetic progressions if $\overline{d}(A) > 0$.*

2 years later Harry (Hillel) Furstenberg gave a new, dynamical, proof of Szemerédi's Theorem. Consequently, our story takes a turn into the world of measure preserving systems and ergodic theory.

Definition 1.1.7. *A* ***measure preserving system (m.p.s.)*** $(X, \mathcal{B}, \mu, T)$ *is a probability space $(X, \mathcal{B}, \mu)$ along with a measurable transformation $T : X \to X$ satisfying $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$.*

[1] Interestingly, if one allows for a broader interpretation of "density analogue", then such a generalization of Schur's theorem is proven in [Ber86].

Furstenberg proved the following multiple recurrence theorem that is equivalent to Szemerédi's theorem.

Theorem 1.1.8 (Theorem 1.4 in [Fur77])**.** *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and $B \in \mathcal{B}$ with $\mu(B) > 0$. For any integer $k > 1$ there exists $n \in \mathbb{N}$ for which*

$$\mu(B \cap T^{-n}B \cap T^{-2n}B \cap \cdots \cap T^{-(k-1)n}B) > 0. \tag{1.2}$$

To derive Szemerédi's theorem from the multiple recurrence theorem, Furstenberg introduced a powerful correspondence principle that now bears his name (Theorem 1.1 in [Fur77]). We provide a variant of Furstenberg's correspondence principle that is better suited for our discussion.

Theorem 1.1.9 (Theorem 1.8 in [Ber96])**.** *If $A \subset \mathbb{N}$ is such that $\overline{d}(A) > 0$, then there exists a m.p.s. $(X, \mathcal{B}, \mu, T)$ and some $E \in \mathcal{B}$ with $\mu(E) = \overline{d}(A)$, such that for any $k \in \mathbb{N}$ and any $n_1, \cdots, n_k \in \mathbb{N}$ we have*

$$\overline{d}(A \cap (A - n_1) \cap \cdots \cap (A - n_k)) \geq \mu(E \cap T^{-n_1}E \cap \cdots \cap T^{-n_k}E). \tag{1.3}$$

It is also worth mentioning that Furstenberg's multiple recurrence theorem is a generalization of a classical recurrence theorem of Henri Poincaré. Poincaré proved Theorem 1.1.10 while studying stability in celestial mechanics (see Section 8 of [Poi90] or [Ber00]).

Theorem 1.1.10 (Poincaré recurrence)**.** *For any m.p.s. $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ for which $\mu(A \cap T^{-n}A) > 0$.*

We observe that the Poincaré recurrence theorem has the following combinatorial analogue: If $A \subset \mathbb{N}$ is such that $\overline{d}(A) > 0$, then there exists $n \in \mathbb{N}$ for which $\overline{d}(A \cap (A - n)) > 0$. Both results can be proven by using the pigeonhole principle. Furstenberg and András Sárközy independently proved refinements of Poincaré's recurrence theorem and its combinatorial analogue by showing that the return time n can be taken to be a perfect square.

Theorem 1.1.11 (Proposition 1.3 in [Fur77])**.** *For any m.p.s. $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ for which $\mu(A \cap T^{-n^2}A) > 0$.*

Theorem 1.1.12 (cf. [Sár78])**.** *If $A \subset \mathbb{N}$ is such that $\overline{d}(A) > 0$, then there exists $n \in \mathbb{N}$ for which $\overline{d}(A \cap (A - n^2)) > 0$.*

Furstenberg's dynamical approach to Szemeredi's theorem spawned the field that is now known as Ergodic Ramsey Theory. This has resulted in a multitude of results about recurrence whose combinatorial analogues still do not have an independent proof. As an example, we mention a special case of a result of Vitaly Bergelson and Alexander Leibman that is a common refinement of the multiple recurrence theorem and the Furstenberg-Sárközy theorem.

Theorem 1.1.13 (cf. [BL96]). *Polynomial Szemerédi Theorem.*

(i) If $(X, \mathcal{B}, \mu, T)$ is a m.p.s. and $\{p_i(x)\}_{i=1}^k \quad \mathbb{Z}[x]$ is a family of polynomials satisfying $p(0) = 0$, then for any $E \quad \mathcal{B}$ with $\mu(E) > 0$ there exists $n \quad \mathbb{N}$ for which

$$\mu(E \cap T^{-p_1(n)}E \cap \cdots \cap T^{-p_k(n)}E) > 0. \tag{1.4}$$

(ii) If $A \quad \mathbb{N}$ is such that $\overline{d}(A) > 0$ and $\{p_i(x)\}_{i=1}^k \quad \mathbb{Z}[x]$ is a family of polynomials satisfying $p(0) = 0$, then there exists $n \quad \mathbb{N}$ for which

$$\overline{d}(A \cap (A - p_1(n)) \cap \cdots \cap (A - p_k(n))) > 0. \tag{1.5}$$

Many novel tools were needed for the proofs of the multiple recurrence theorem, the Furstenberg-Sárközy theorem, and the polynomial Szemerédi theorem. One of those tools is the van der Corput difference theorem (vdCdt) and its many variations. Johannes Gualtherus van der Corput introduced the first form of the vdCdt in [Cor31]. To state the first vdCdt we first require a definition.

Definition 1.1.14. *A sequence $(x_n)_{n=1}^\infty \quad [0,1]$ is **uniformly distributed** if for any $0 \le a < b \le 1$ we have*

$$\lim_{N \to \infty} \frac{1}{N} |\{1 \le n \le N \mid x_n \quad (a,b)\}| = b - a. \tag{1.6}$$

Theorem 1.1.15 (vdCdt, Theorem 1.3.1 in [KN74]). *If $(x_n)_{n=1}^\infty \quad [0,1]$ is a sequence for which $(x_{n+h} - x_n)_{n=1}^\infty \pmod 1$ is uniformly distributed for all $h \quad \mathbb{N}$, then $(x_n)_{n=1}^\infty$ is uniformly distributed.*

A classical application of vdCdt is to show that for $\alpha \quad \mathbb{R} \setminus \mathbb{Q}$ the sequence $(n^2\alpha)_{n=1}^\infty$ (mod 1) is uniformly distributed in $[0,1]$, which is a fact used in Furstenberg's proof of Theorem 1.1.11. In [Ber87] Bergelson introduced several variations of vdCdt for bounded sequences of vectors in a Hilbert space, one of which we introduce below.

Theorem 1.1.16 (Theorem 1.4 in [Ber87]). *Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^\infty \quad \mathcal{H}$ a bounded sequence. If*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0 \tag{1.7}$$

for all $h \in \mathbb{N}$, *then*

$$\lim_{N\to\infty} \|\frac{1}{N} \sum_{n=1}^{N} x_n\| = 0. \tag{1.8}$$

The subject of Chapter 2 is to prove new variations of Theorems 1.1.15 and 1.1.16, and to show that these new variations are connected to the ergodic hierarchy of mixing properties of a m.p.s. (cf. Definition 2.5.2).

There are many results about recurrence other than Theorem 1.1.11 that are proven using tools from the theory of uniform distribution. In Chapter 4 we investigate a curious generalization of the notion of uniform distribution which we call *uniform symmetric distribution* (Definition 4.1.4) and show that it too is associated with the study of recurrence (cf. Theorem 4.2.1). We also prove a version of vdCdt using uniform symmetric distribution (see Theorem 4.2.9).

In order to connect Chapter 3 to our story we will now discuss some other ergodic theorems. We begin with the mean ergodic theorem of John von Neumann from [Neu32].

Theorem 1.1.17. *Let* $(X, \mathcal{B}, \mu, T)$ *be a m.p.s., let* $U_T : L^2(X,\mu) \to L^2(X,\mu)$ *be the unitary operator given by* $U_T f := f \circ T$*, and let* P_T *denote the orthogonal projection onto the closed subspace*

$$I := \{g \in L^2(X,\mu) \mid U_T g = g\}. \tag{1.9}$$

Then for any $f \in L^2(X,\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U_T^n f = P_T f, \tag{1.10}$$

with convergence taking place in the norm topology.

The mean ergodic theorem played a part in the proofs of many of the multiple recurrence theorems mentioned so far. To further illustrate this connection, we observe that Furstenberg derived Theorem 1.1.8 from a stronger result that we will now state.

Theorem 1.1.18 (Theorem 11.13 in [Fur77]). *If $(X, \mathcal{B}, \mu, T)$ is an ergodic*[2] *m.p.s. and $f \in L^\infty(X,\mu)$ is such that $f \geq 0$ and f is not a.e. 0, then for any $k \in \mathbb{N}$,*

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M+1}^{N} \int_X f \cdot U_T^n f \cdots U_T^{n(k-1)} f d\mu > 0. \tag{1.11}$$

Another classical ergodic theorem was proven by George David Birkhoff in [Bir31].

Theorem 1.1.19. *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s., and let $f \in L^1([0,1],\mu)$. For a.e. $x \in X$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = f^*(x), \tag{1.12}$$

where $f^*(x) \in L^1(X,\mu)$ is such that $f^*(Tx) = f^*(x)$ for a.e. $x \in X$ and $\int_X f^* d\mu = \int_X f d\mu$.

While Birkhoff's ergodic theorem is not as commonly used as von Neumann's ergodic theorem when studying recurrence, it is still a subject of great interest in ergodic theory. The subject of Chapter 3 is to prove strengthenings of Birkhoff's ergodic theorem when additional assumptions are made about which level of the ergodic hierarchy of mixing the m.p.s. $(X, \mathcal{B}, \mu, T)$ belongs to.

In Chapter 5 we study notion of van der Corput sets introduced by Teturo Kamae and Michel Mendès France in [KMF78].

Definition 1.1.20. $R \subseteq \mathbb{N}$ *is a* ***van der Corput set (vdC set)*** *if for any $(x_n)_{n=1}^\infty \subseteq [0,1]$ for which $(x_{n+h} - x_n)_{n=1}^\infty$ is uniformly distributed for all $h \in R$ we have that $(x_n)_{n=1}^\infty$ is uniformly distributed.*

Kamae and Mendès France showed that vdC sets are related to recurrence while Imre Ruzsa showed in [Ruz84] that vdC sets have connections with harmonic analysis. It was also shown by Marina Ninčević, Braslav Rabar, and Siniša Slijepčević in [NRS12] that vdC sets are connected to a notion of recurrence in Hilbert spaces that they refer to as operator recurrence (see Definition 5.1.4(v)). We will prove similar results to those just mentioned for some variations of vdC sets that are introduced in [BL08]. We will also show that vdC sets have connections with the results of Chapter 2.2 as well as the notion of uniform symmetric distribution.

[2] A m.p.s. $(X, \mathcal{B}, \mu, T)$ is **ergodic** if the $A \in \mathcal{B}$ for which $\mu(T^{-1}A \Delta A) = 0$ also satisfy $\mu(A) \in \{0,1\}$.

1.2 Overview

In Chapter 2 we investigate variations of vdCdt in Hilbert spaces as well as the theory of uniform distribution and show that they are connected to the ergodic hierarchy of mixing properties. In Chapter 3 we show how Birkhoff's ergodic theorem can be strengthened when additional assumptions are made about the mixing properties of the underlying m.p.s. $(X, \mathcal{B}, \mu, T)$. We do this in the more general set up of Hilbert space-valued functions rather than complex-valued functions so that the connections with Chapter 2 become clearer. In Chapter 4 we introduce and study the notion of uniform symmetric distribution. We also show many ergodic theorems involving uniform distribution have analogues involving uniform symmetric distribution. In Chapter 2 we add to the known list of equivalent formulations of van der Corput sets. We also answer some open questions from the literature regarding van der Corput sets, and we show how they connect to the results from Chapters 2 and 4. In Chapter 6 we give an almost complete classification of those $m, n \in \mathbb{N}$ and $a, b, c \in \mathbb{Z} \setminus \{0\}$ for which the equation $ax + by = cw^m z^n$ is partition regular over $\mathbb{Z} \setminus \{0\}$. Many tools from Ramsey theory and number theory are needed to prove the aforementioned result. In particular, we prove a partial generalization of the Grunwald-Wang theorem, and we extend many classical results about ultrafilters over $\mathbb{N}$ to ultrafilters over an integral domain R.

CHAPTER 2

ENHANCEMENTS OF VAN DER CORPUT'S DIFFERENCE THEOREM AND CONNECTIONS TO THE ERGODIC HIERARCHY OF MIXING PROPERTIES

2.1 Introduction

In [Cor31] J. G. van der Corput proved Theorem 2.1.1 which is now known as van der Corput's Difference Theorem (henceforth abbreviated as vdCDT).

Theorem 2.1.1 (Theorem 1.3.1 in [KN74]). *If* $(x_n)_{n=1}^{\infty}$ $[0,1]$ *is a sequence for which* $(x_{n+h} - x_n)_{n=1}^{\infty}$ (mod 1) *is uniformly distributed for all* h N, *then* $(x_n)_{n=1}^{\infty}$ *is uniformly distributed.*

In Ergodic Theory, the following Hilbertian analogues of Theorem 2.1.1 were introduced by V. Bergelson in [Ber87] and are of great use.

Theorem 2.1.2. *Let H be a Hilbert space and* $(x_n)_{n=1}^{\infty}$ *H a bounded sequence of vectors.*

(i) If for every h N *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_{n+h}, x_n = 0, \text{ then } \lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}x_n|| = 0. \tag{2.1}$$

(ii) If

$$\lim_{h\to\infty}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} x_{n+h}, x_n \right| = 0, \text{ then } \lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}x_n|| = 0. \tag{2.2}$$

(iii) If

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} x_{n+h}, x_n \right| = 0, \text{ then } \lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}x_n|| = 0. \tag{2.3}$$

See [Ber96], [BL96], [FLW06], [EW11], and [Fur81] for some examples of applications of parts (i), (ii), and (iii) of Theorem 2.1.2 in Ergodic Theory, and see [BM16] for a survey of modern developments regarding vdCDT and its many variations. We point out to the reader that item (iii) of Theorem 2.1.2 implies items (i) and (ii), which begs the question of why one would ever use items (i) and (ii). To answer this question, we will show a correspondence between the variants of vdCDT appearing in Theorem 2.1.2 and the ergodic hierarchy of mixing. In particular, we will show that Theorem 2.1.2(i) corresponds to Lebesgue spectrum (cf. Corollary 2.2.17), Theorem 2.1.2(ii) corresponds to strong mixing (cf. Corollary 2.2.15), and Theorem 2.1.2(iii) corresponds to weak mixing (cf. Corollary 2.2.11). Using this correspondence we will prove new variants of vdCDT corresponding to ergodicity (cf. Corollary 2.2.9) and mild mixing (cf. Corollary 2.2.13). We remark that a connection between forms of vdCDT that do not use Cesàro averages and the ergodic hierarchy of mixing was already observed in [Tse16]. We also remark that our methods apply when taking averages over Følner sequences $(\Psi_n)_{n=1}^{\infty}$ other than $([1,N])_{N=1}^{\infty}$, but we do not pursue this level of generality for the sake of concreteness.

In Section 2.2 we show how one can use a given Hilbert space H to construct a new Hilbert space H whose elements are sequences of vectors from H. We then define various classes of mixing sequences of vectors from H associated to various levels of the ergodic hierarchy of mixing (cf. Definition 2.2.5). We show that a sequence of vectors in H is one of our mixing sequences if and only if it is a mixing element of H (cf. Definition 2.2.4) under a certain unitary operator induced by the left shift. We are then able to prove the results that are mentioned in the previous paragraph by showing that different variations of vdCDT produce different classes of mixing sequences.

In Section 2.3 we begin by showing that any of the (possibly unbounded) mixing sequences appearing in Definition 2.2.5 have Cesàro averages that converge strongly to 0 (cf. Lemma 2.3.1(iii)-(iv)), which is what allows us to conclude that that the results in Section 2.2 are generalizations of the results appearing in Theorem 2.1.2. We then proceed to analyze additional properties of these mixing sequences such as which of their weighted Cesàro averages converge strongly to 0 (cf. Theorem 2.3.9).

In Section 2.4 we obtain applications to the Theory of Uniform Distribution by strengthening the result of Theorem 2.1.1, and by producing 4 new variants of vdCDT for uniform distribution that correspond to different levels of the ergodic hierarchy of mixing. 2 of our new vdCDTs (cf. Corollaries 2.4.18 and 2.4.22) have a correspondence to Theorem 2.1.2(ii) and Theorem 2.1.2(iii) like that of Theorem 2.1.1 and Theorem 2.1.2(i). Similar to what we did in Section 2.2, we define various classes of mixing sequences in $[0,1]$ (cf. Definition 2.4.6) and show that each of our new vdCDTs for uniform distribution produces one of these mixing sequences. We also show that each of these mixing sequences are uniformly distributed along many subsequences (cf. Theorem 2.4.9). Interestingly, we are unable to find a clear correspondence between Theorem 2.1.2 and mixing sequences in $[0,1]$ (cf. Remark 2.4.25), but we are able to show that $(x_{n+h}, x_n)_{n=1}^{\infty} \in [0,1]^2$ is uniformly distributed for all $h \in \mathbb{N}$ if and only if $(x_n)_{n=1}^{\infty} \in [0,1]$ is sufficiently mixing (cf. Theorem 2.4.26).

In Section 2.5 we review the Ergodic Hierarchy of Mixing for measure preserving systems (cf. Definition 2.5.2) and show how the mixing sequences from Definition 2.2.5 can be used to characterize various levels of this hierarchy (cf. Theorem 2.5.3). We then obtain applications to recurrence in measure preserving systems and give a partial answer to a question of N. Frantzikinakis from [Fra22]. We conclude this with a generalization of the main result N. Frantzikinakis, E. Lesigne, and M. Wierdl from [FLW06].

In Section 2.6 we compare the notions of mixing sequences that we introduce in Definition 2.2.5 with those introduced in [BB86]. Section 2.7 is a review of basic definitions and properties of IP-sets and IP convergence, which are notions that will be used throughout the paper when discussing mild mixing and rigidity. We also review some properties of ultrafilters, which will only be used in the proof of Lemma 2.3.6(iii).

2.2 Extensions of van der Corput's Difference Theorem in Hilbert Spaces

We will now view van der Corput's Difference Theorem from a Hilbertian point of view that is different from Theorem 2.1.2. First, we need to construct the new Hilbert space H that we will be working with from a given Hilbert space H. We will let $||\cdot||$ and $\langle \cdot, \cdot \rangle$ denote the norm and inner product on H while letting $||\cdot||_H$ and $\langle \cdot, \cdot \rangle_H$ denote the norm and inner product on H. We denote the collection of square averageable sequences by

$$SA(H) := \{(f_n)_{n=1}^{\infty} \subseteq H \mid \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} ||f_n||^2 < \infty\}, \tag{2.4}$$

and we denote the collection of uniformly bounded sequences by

$$UB(H) := \{(f_n)_{n=1}^{\infty} \subseteq H \mid \sup_{n\in\mathbb{N}} ||f_n|| < \infty\}. \tag{2.5}$$

Let $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \in SA(H)$ and observe that

$$\limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{N}\langle f_n, g_n\rangle\right| \le \limsup_{N\to\infty}\frac{1}{N}\sum ||f_n||\cdot||g_n|| \tag{2.6}$$
$$\le\left(\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||f_n||^2\right)^{\frac{1}{2}}\left(\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||g_n||^2\right)^{\frac{1}{2}} < \infty.$$

It follows that we may use diagonalization to construct an increasing sequence of positive integers $(N_q)_{q=1}^{\infty}$ for which

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle x_{n+h}, y_n\rangle \tag{2.7}$$

exists whenever $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \in \{(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}\}$ and $h \in \mathbb{N}$. We can now construct a new Hilbert space $H = H((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ from $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}$ and $(N_q)_{q=1}^{\infty}$ as follows. For all $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \in \{(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}\}$ and $h \in \mathbb{N}$, we define

$$\langle (x_{n+h})_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}\rangle_{H} = \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle x_{n+h}, y_n\rangle. \tag{2.8}$$

We see that $\langle\cdot,\cdot\rangle_{H}$ is a sesquilinear form on $H = \text{Span}_{\mathbb{C}}(\{(f_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty} \cup \{(g_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty})$. Letting

$$H = \{(e_n)_{n=1}^{\infty} \in SA(H) \mid \forall\epsilon > 0\ \exists(h_n(\epsilon))_{n=1}^{\infty} \in H \text{ s.t. } \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||e_n - h_n(\epsilon)||^2 < \epsilon\}, \text{ and} \tag{2.9}$$

$$S = \{(x_n)_{n=1}^{\infty} \in SA(H) \mid \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||x_n||^2 = 0\}, \tag{2.10}$$

we see that H/S is a pre-Hilbert space. We will soon see that H is sequentially closed under the topology induced by $\langle\cdot,\cdot\rangle_{H}$ (cf. Theorem 2.2.1), so we define $H((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = H/S$. We call $H((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ the Hilbert space induced by $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, and we may write H in place of $H((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ if $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is understood from the context.

For $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \in SA(H)$ we say that $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a **permissible triple** if $H((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is well defined. Whenever we write $H((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty},$

$(N_q)_{q=1}^{\infty})$, it is implicitly assumed that $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple. We say that $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a **weakly permissible triple** if

$$\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, g_n \tag{2.11}$$

exists for every $h \quad \mathsf{N}$. Given $(x_n)_{n=1}^{\infty} \quad H$ for which $(x_n)_{n=1}^{\infty} \quad H \quad$, we may view $(x_n)_{n=1}^{\infty}$ as an element of H by identifying $(x_n)_{n=1}^{\infty}$ with its equivalence class in $H \quad /S$.

Our applications in sections 2.4 and 2.5 will only require results about $UB(H)$. Nonetheless, we are required to work with $SA(H)$ so that we may construct the Hilbert space H . To see that this is the case, let us consider the collection of sequences $\{\xi_m\}_{m=1}^{\infty} \quad UB(\mathsf{C})$ given by $\xi_{m,n} = 0$ if $n \equiv 0 \pmod{m^6}$ and $\xi_{m,n} = m$ if $n \equiv 0 \pmod{m^6}$. We see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \xi_{m_1,n+h}, \xi_{m_2,n} \tag{2.12}$$

exists for all $m_1, m_2, h \quad \mathsf{N}$, so $\{\xi_m\}_{m=1}^{\infty} \quad H \quad = H \; (\xi_1, \xi_2, (n)_{n=1}^{\infty})$. We also see that $||\xi_n||_H \; = \frac{1}{n^2}$, so $\xi := \sum_{n=1}^{\infty} \xi_n \quad H \; \setminus \mathrm{UB}(H)$.

We will now verify that H is a Hilbert space by verifying that it is complete.

Theorem 2.2.1. *Let H be a Hilbert space and $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \quad SA(H)$. Let $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ be a permissible triple and $H \; = H \; ((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. If $\{(\xi_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty} \quad H$ is a Cauchy sequence, then there exists $(\xi_n)_{n=1}^{\infty} \quad H$ for which*

$$\lim_{m\to\infty} \left(\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||\xi_{n,m} - \xi_n||^2 \right) = 0. \tag{2.13}$$

In particular, H is a Hilbert space.

Proof. We proceed by modifying the proof of the main result in section §2 of chapter II of [BF45]. Let $(\epsilon_m)_{m=1}^{\infty}$ be a sequence of real numbers tending to 0 for which

$$\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||\xi_{n,m} - \xi_{n,k}||^2 < \epsilon_m \tag{2.14}$$

whenever $k \geq m$. By induction, let $T_0 = N_0 = 0$ and let $(T_m)_{m=1}^{\infty} \quad \mathsf{N}$ be such that conditions (i)-(iii) below hold.

(i) For every $m \geq 1$, every $k \geq m$, and every $T \geq T_k$

$$\frac{1}{N_T} \sum_{n=1}^{N_T} ||\xi_{n,k} - \xi_{n,m}||^2 < \epsilon_m. \tag{2.15}$$

(ii) For every $m \geq 1$ and every $k \geq m$

$$\frac{1}{N_{T_k} - N_{T_{k-1}}} \sum_{n=N_{T_{k-1}}+1}^{N_{T_k}} ||\xi_{n,k} - \xi_{n,m}||^2 < \epsilon_m. \tag{2.16}$$

(iii) For every $m \geq 1$

$$\frac{1}{N_{T_m}} \sum_{j=1}^{m-1} \sum_{n=N_{T_{j-1}}+1}^{N_{T_j}} ||\xi_{n,j} - \xi_{n,m}||^2 < \epsilon_m. \tag{2.17}$$

Now let us define $(\xi_n)_{n=1}^{\infty}$ by $\xi_n = \xi_{n,m}$ where m is such that $N_{T_{m-1}} < n \leq N_{T_m}$. To conclude the proof, we note that for $m \geq 1$, $k > m$, and $T_{k-1} < T \leq T_k$ we have

$$\begin{aligned}
&\sum_{n=1}^{N_T} ||\xi_{n,m} - \xi_n||^2 \qquad (2.18)\\
&= \sum_{j=1}^{m-1} \sum_{n=N_{T_{j-1}}+1}^{N_{T_j}} ||\xi_{n,j} - \xi_{n,m}||^2 + \sum_{j=m}^{k-1} \sum_{n=N_{T_{j-1}}+1}^{N_{T_j}} ||\xi_{n,m} - \xi_n||^2 + \sum_{n=N_{T_{k-1}}+1}^{N_T} ||\xi_{n,m} - \xi_n||^2\\
&\leq N_{T_m}\epsilon_m + \sum_{j=m}^{k-1} (N_{T_j} - N_{T_{j-1}})\epsilon_m + \sum_{n=1}^{N_T} ||\xi_{n,k} - \xi_{n,m}||^2\\
&\leq N_{T_{k-1}}\epsilon_m + N_T\epsilon_m \leq 2N_T\epsilon_m.
\end{aligned}$$

□

Lemma 2.2.2. *Let H be a Hilbert space and $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \quad SA(H)$. If $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple and $H \ = H \ ((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, then for all $M > 0$*

$$B_M := \{(x_n)_{n=1}^{\infty} \quad H \ \mid ||x_n|| \leq M \quad n \quad \mathrm{N}\} \tag{2.19}$$

is compact in the weak topology of H .

Proof. Since

$$B(M) := \{(x_n)_{n=1}^{\infty} \quad H \mid \|(x_n)_{n=1}^{\infty}\|_{H} \leq M\} \tag{2.20}$$

is known to be a compact set as a consequence of the Banach-Alaoglu Theorem and B_M $B(M)$, it suffices to show that B_M is closed in the weak topology. To this end, let $(x_n)_{n=1}^{\infty}$ B_M^c be arbitrary. We see that there exists $\epsilon > 0$ for which

$$A := \{n \quad \mathrm{N} \mid \|x_n\| > M + \epsilon\} \tag{2.21}$$

satisfies $\limsup_{q\to\infty} \frac{1}{N_q}|[1, N_q] \cap A| > 0$, otherwise the equivalence class of $(x_n)_{n=1}^{\infty}$ in H would contain an element of B_M. Let $(M_q)_{q=1}^{\infty}$ be such that $\lim_{q\to\infty} \frac{1}{M_q}|[1, M_q] \cap A| > 0$, and let $(M_q)_{q=1}^{\infty}$ be a subsequence of $(M_q)_{q=1}^{\infty}$ for which

$$F(y_n)_{n=1}^{\infty} := \lim_{q\to\infty} \frac{1}{M_q} \sum_{n=1}^{M_q} y_n, \mathbb{1}_A(n)x_n \tag{2.22}$$

exists for all $(y_n)_{n=1}^{\infty}$ H. We see that F uniquely extends to a continuous functional on all of H. We now see that if $(y_n)_{n=1}^{\infty}$ B_M then

$$\begin{aligned}
&|F((x_n)_{n=1}^{\infty}| - |F(y_n)_{n=1}^{\infty})| \\
&= \lim_{q\to\infty} \frac{1}{M_q}\left|\sum_{n=1}^{M_q} x_n, \mathbb{1}_A(n)x_n\right| - \lim_{q\to\infty} \frac{1}{M_q}\left|\sum_{n=1}^{M_q} y_n, \mathbb{1}_A(n)x_n\right| \\
&\geq \lim_{q\to\infty} \frac{1}{M_q}\sum_{n=1}^{M_q} \mathbb{1}_A(n)\|x_n\|^2 - \lim_{q\to\infty} \frac{1}{M_q}\sum_{n=1}^{M_q} \mathbb{1}_A(n)\|x_n\| \cdot \|y_n\| \\
&= \lim_{q\to\infty} \frac{1}{M_q}\sum_{n=1}^{M_q} \mathbb{1}_A(n)\|x_n\|(\|x_n\| - \|y_n\|) \geq \lim_{q\to\infty} \frac{1}{M_q}\sum_{n=1}^{M_q} \mathbb{1}_A(n)(M+\epsilon)\epsilon > 0.
\end{aligned} \tag{2.23}$$

□

Remark 2.2.3. Let H be a Hilbert space. Let $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}$ H and $(N_q)_{q=1}^{\infty}$ be such that $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple. The left shift $S : H \to H$ defined by $S((x_n)_{n=1}^{\infty}) = (x_{n+1})_{n=1}^{\infty}$ for all $(x_n)_{n=1}^{\infty}$ H extends to an operator on H that we again denote by S. Since $S((x_n)_{n=1}^{\infty}), S((y_n)_{n=1}^{\infty})_{H} = (x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}{}_{H}$ for all $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ H, we see that S is a unitary operator on H. This allows us to classify sequences in H that correspond to elements of (H, S) from different levels of the hierarchy of mixing. Since we naturally have that B H and that S preserves uniform boundedness, we may also consider S as a unitary operator on B. To this end, we begin by recalling some vocabulary regarding the ergodic hierarchy of mixing.

Definition 2.2.4. *Let H be a Hilbert space, $U : H \to H$ a unitary operator, and $f \in H$.*

*(i) f is an **ergodic element of** (H, U) if for all $g \in H$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle U^n f, g \rangle = 0. \tag{2.24}$$

*(ii) f is a **weakly mixing element of** (H, U) if for all $g \in H$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U^n f, g \rangle| = 0. \tag{2.25}$$

*(iii) f is a **mildly mixing element of** (H, U) if for all $g \in H$ we have*

$$IP - \lim_{n \to \infty} \langle U^n f, g \rangle = 0. \tag{2.26}$$

*(iv) f is a **strongly mixing element of** (H, U) if for all $g \in H$ we have*

$$\lim_{N \to \infty} \langle U^n f, g \rangle = 0. \tag{2.27}$$

If H and U are understood from the context, then we may omit (H, U) and say (for example) that f is a weakly mixing element.

Definition 2.2.5. *Let H be a Hilbert space and $(f_n)_{n=1}^{\infty} \subseteq SA(H)$.*

*(i) $(f_n)_{n=1}^{\infty}$ is a **completely ergodic sequence**[1] if for all weakly permissible triple of the form $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, we have*

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle f_{n+h}, g_n \rangle = 0. \tag{2.28}$$

*(ii) $(f_n)_{n=1}^{\infty}$ is a **nearly weakly mixing sequence**[1] if for all weakly permissible triple of the form $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, we have*

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \left| \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle f_{n+h}, g_n \rangle \right| = 0. \tag{2.29}$$

[1] See the appendix for a comparison of our definitions of completely ergodic sequences, nearly weakly mixing sequences, and nearly strongly mixing sequences with the definitions of ergodic sequences, weakly mixing sequences, and strongly mixing sequences given in [BB86].

(iii) $(f_n)_{n=1}^{\infty}$ *is a* ***nearly mildly mixing sequence*** *if for all weakly permissible triple of the form* $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$*, we have*

$$IP - \lim_{h \to \infty} \left| \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, g_n \right| = 0. \tag{2.30}$$

(iv) $(f_n)_{n=1}^{\infty}$ *is a* ***nearly strongly mixing sequence***[1] *if for all weakly permissible triple of the form* $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$*, we have*

$$\lim_{h \to \infty} \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, g_n = 0. \tag{2.31}$$

(v) $(f_n)_{n=1}^{\infty}$ *is a* ***nearly orthogonal sequence*** *if for all permissible triple of the form* $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$*, we have*

$$\sum_{h=0}^{\infty} \left| \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, g_n \right|^2 \le \left(\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||g_n||^2 \right) \left(\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||f_n||^2 \right). \tag{2.32}$$

Remark 2.2.6. We see that if $(f_n)_{n=1}^{\infty}$ H is a nearly weakly (strongly) mixing sequence if and only if for all permissible triple of the form $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, $(f_n)_{n=1}^{\infty}$ is a nearly weakly (strongly) mixing element of $(H\ ((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}), S)$. Furthermore, we will see that if $U : H \to H$ is a unitary operator, then for all f H, $(U^n f)_{n=1}^{\infty}$ is a nearly weakly (strongly) mixing sequence if and only if U is a nearly weakly (strongly) mixing unitary operator.

Remark 2.2.7. Nearly orthogonal sequences display mixing properties similar to that of an orthornomal set $\{e_n\}_{n=1}^{\infty}$ in a Hilbert space H. We recall that for all x H, we have

$$\sum_{n=1}^{\infty} | x, e_n |^2 \le M||x||^2, \tag{2.33}$$

where M is an upper bound of the sequence $\{||e_n||\}_{n=1}^{\infty}$. We now see that if $(f_n)_{n=1}^{\infty}$ H is a nearly orthogonal sequence, then for all permissible triple of the form $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ and $H = H\ ((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, we have

$$\sum_{h=1}^{\infty} S^h((f_n)_{n=1}^{\infty}), (g_n)_{n=1}^{\infty}\ {}_H \le M||(g_n)_{n=1}^{\infty}||_H^2 . \tag{2.34}$$

It follows that $(f_n)_{n=1}^{\infty}$ exhibits mixing properties similar to that of an orthonormal set of vectors, however, it should be noted that there exist sets of vectors that are not orthonormal, but are still nearly orthogonal. For example, if $\{e_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded orthonormal set as before, then the sequence $(f_n)_{n=1}^{\infty}$ given by $f_n = \frac{1}{2}(e_n + e_{n+1})$ satisfies equation (2.34) with $M = 2$, but is not an orthogonal set.

We are now ready to state and prove some of the main results of this paper.

Theorem 2.2.8. *Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^{\infty} \in SA(\mathcal{H})$. $(f_n)_{n=1}^{\infty}$ is a completely ergodic sequence if and only if for all permissible triple of the form $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have*

$$\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} \langle f_{n+h}, f_n \rangle = 0. \tag{2.35}$$

Proof. If $(f_n)_{n=1}^{\infty}$ is a completely ergodic sequence, then the desired result follows immediately. Now let us show that $(f_n)_{n=1}^{\infty}$ is a completely ergodic sequence if equation (2.35) is satisfied. Let $(g_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ be such that $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple. By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, so may define $\mathscr{H} = \mathscr{H}((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Let $S : \mathscr{H} \to \mathscr{H}$ denote the unitary operator induced by the left shift. Let $\mathscr{H} = \mathscr{I} \oplus \mathscr{I}^{\perp}$, where $\mathscr{I} := \{\xi \in \mathscr{H} \mid S\xi = \xi\}$ and $\mathscr{I}^{\perp} = c\ell(\{\xi - S\xi \mid \xi \in \mathscr{H}\})$ (cf. Theorem 2.21 in [EW11]). Let $(f_n)_{n=1}^{\infty} = \xi_1 + \xi_2$ with $\xi_1 \in \mathscr{I}$ and $\xi_2 \in \mathscr{I}^{\perp}$. Noting that $\mathscr{I}$ and $\mathscr{I}^{\perp}$ are invariant under S and that equation (2.35) can be restated as

$$0 = \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \langle (f_{n+h})_{n=1}^{\infty}, (f_n)_{n=1}^{\infty} \rangle_{\mathscr{H}} \tag{2.36}$$

$$= \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \langle S^h (f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty} \rangle_{\mathscr{H}} \tag{2.37}$$

$$= \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \Big(\langle S^h \xi_1, \xi_1 \rangle_{\mathscr{H}} + \langle S^h \xi_2, \xi_2 \rangle_{\mathscr{H}} \Big) \tag{2.38}$$

$$= ||\xi_1||_{\mathscr{H}}^2 + \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \langle S^h \xi_2, \xi_2 \rangle_{\mathscr{H}} = ||\xi_1||_{\mathscr{H}}^2, \tag{2.39}$$

we see that $(f_n)_{n=1}^{\infty} \in \mathscr{I}^{\perp}$. It follows that

$$0 = \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} S^h((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \ \mathcal{H} \tag{2.40}$$

$$= \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, g_n \ . \tag{2.41}$$

Since $(g_n)_{n=1}^{\infty}$ and $(N_q)_{q=1}^{\infty}$ were both arbitrary, we see that $(f_n)_{n=1}^{\infty}$ is a completely ergodic sequence. □

Our next corollary can be seen as a generalization of the variant of vdCdt appearing in page 445 of [BL02].

Corollary 2.2.9. *Let $\mathcal{H}$ be a Hilbert space and let $(x_n)_{n=1}^{\infty}$ $\mathcal{H}$ be a bounded sequence. If*

$$\limsup_{H \to \infty} \limsup_{N \to \infty} \left| \frac{1}{NH} \sum_{(h,n) \ [1,H]\times[1,N]} x_{n+h}, x_n \ \right| = 0, \tag{2.42}$$

then $(x_n)_{n=1}^{\infty}$ is a completely ergodic sequence.

Proof. Equation (2.42) implies equation (2.35) for any sequence $(N_q)_{q=1}^{\infty}$, and all bounded sequences are contained in $SA(\mathcal{H})$. □

Theorem 2.2.10. *Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^{\infty}$ $SA(\mathcal{H})$. $(f_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence if and only if for all permissible triple of the form $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have*

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \left| \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, f_n \ \right| = 0. \tag{2.43}$$

Proof. If $(f_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence, then the desired result follows immediately. Now let us show that $(f_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence if equation (2.43) is satisfied. Let $(g_n)_{n=1}^{\infty}$ $\mathcal{H}$ and $(N_q)_{q=1}^{\infty}$ $\mathbb{N}$ be such that $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple. By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, so may define $\mathcal{H} = \mathcal{H}((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Let $S : \mathcal{H} \to \mathcal{H}$ denote the unitary operator induced by the left shift. From equation (2.43) we see that

$$\lim_{H \to \infty} \sum_{h=1}^{H} | \ S^h((f_n)_{n=1}^{\infty}), (f_n)_{n=1}^{\infty} \ \mathcal{H} \ | = 0, \tag{2.44}$$

so $(f_{n+h})_{n=1}^{\infty}$ is a weakly mixing element of $(\mathcal{H}, U)$ by Lemma 3 of [Ŕ58]. It follows that

$$0 = \lim_{H\to\infty} \sum_{h=1}^{H} | \langle S^h((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}\rangle_{\mathcal{H}} | = \lim_{H\to\infty} \sum_{h=1}^{H} | \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle f_{n+h}, g_n\rangle |. \tag{2.45}$$

Since $(g_n)_{n=1}^{\infty}$ and $(N_q)_{q=1}^{\infty}$ were both arbitrary, we see that $(f_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence. □

Corollary 2.2.11. *Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a bounded sequence. If*

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle f_{n+h}, f_n\rangle \right| = 0, \tag{2.46}$$

then $(f_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

Proof. Equation (2.46) implies equation (2.43) for any sequence $(N_q)_{q=1}^{\infty}$, and all bounded sequences are contained in $SA(\mathcal{H})$. □

Theorem 2.2.12. *Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^{\infty} \in SA(\mathcal{H})$. $(f_n)_{n=1}^{\infty}$ is a nearly mildly mixing sequence if and only if for all permissible triple of the form $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have*

$$IP-\lim_{h\to\infty} \left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle f_{n+h}, f_n\rangle \right| = 0. \tag{2.47}$$

Proof. If $(f_n)_{n=1}^{\infty}$ is a nearly mildly mixing sequence, then the desired result follows immediately. Now let us show that $(f_n)_{n=1}^{\infty}$ is a nearly mildly mixing sequence if equation (2.47) is satisfied. Let $(g_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ be such that $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple. By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, so may define $\mathcal{H} = \mathcal{H}((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Let $S : \mathcal{H} \to \mathcal{H}$ denote the unitary operator induced by the left shift. From equation (2.47) we see that

$$\text{IP}-\lim_{h\to\infty} | \langle S^h((f_n)_{n=1}^{\infty}), (f_n)_{n=1}^{\infty}\rangle_{\mathcal{H}} | = 0, \tag{2.48}$$

so $(f_{n+h})_{n=1}^{\infty}$ is a mildly mixing element of $(\mathcal{H}, U)$ by Lemma 9.24 of [Fur81]. It follows that

$$0 = \text{IP}-\lim_{h\to\infty} | \langle S^h((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}\rangle_{\mathcal{H}} | \tag{2.49}$$

$$= \mathrm{IP} - \lim_{h\to\infty} | \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, g_n \; |. \tag{2.50}$$

Since $(g_n)_{n=1}^{\infty}$ and $(N_q)_{q=1}^{\infty}$ were both arbitrary, we see that $(f_n)_{n=1}^{\infty}$ is a nearly mildly mixing sequence. □

Corollary 2.2.13. *Let H be a Hilbert space and let* $(f_n)_{n=1}^{\infty} \quad H$ *be a bounded sequence. If*

$$IP - \lim_{h\to\infty} \left| \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_{n+h}, f_n \right| = 0, \tag{2.51}$$

then $(f_n)_{n=1}^{\infty}$ *is a nearly mildly mixing sequence.*

Proof. Equation (2.51) implies equation (2.47) for any sequence $(N_q)_{q=1}^{\infty}$, and all bounded sequences are contained in $SA(H)$. □

Theorem 2.2.14. *Let H be a Hilbert space and let* $(f_n)_{n=1}^{\infty} \quad SA(H)$. $(f_n)_{n=1}^{\infty}$ *is a nearly strongly mixing sequence if and only if for all permissible triple of the form* $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ *we have*

$$\lim_{h\to\infty} \lim_{q\to\infty} \left| \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, f_n \right| = 0. \tag{2.52}$$

Proof. If $(f_n)_{n=1}^{\infty}$ is a nearly strongly mixing sequence, then the desired result follows immediately. Now let us show that $(f_n)_{n=1}^{\infty}$ is a nearly strongly mixing sequence if equation (2.52) is satisfied. Let $(g_n)_{n=1}^{\infty} \quad H$ and $(N_q)_{q=1}^{\infty} \quad \mathsf{N}$ be such that $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple. By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, so may define $H = H((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Let $S : H \to H$ denote the unitary operator induced by the left shift. From equation (2.2.15) we see that

$$\lim_{h\to\infty} S^h((f_n)_{n=1}^{\infty}), (f_n)_{n=1}^{\infty} \;_{H} = 0, \tag{2.53}$$

so $(f_n)_{n=1}^{\infty}$ is a strongly mixing element of (H, S) by Lemma 1 of [Ŕ58]. It follows that

$$0 = \lim_{h\to\infty} S^h((f_n)_{n=1}^{\infty}), (g_n)_{n=1}^{\infty} \;_{H} = \lim_{h\to\infty} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f_{n+h}, g_n \; . \tag{2.54}$$

Since $(g_n)_{n=1}^{\infty}$ and $(N_q)_{q=1}^{\infty}$ were both arbitrary, we see that $(f_n)_{n=1}^{\infty}$ is a nearly strongly mixing sequence. □

Corollary 2.2.15. *Let H be a Hilbert space and let $(f_n)_{n=1}^{\infty} \subseteq H$ be a bounded sequence. If*

$$\lim_{h\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N}\sum_{n=1}^{N} \langle f_{n+h}, f_n\rangle \right| = 0, \tag{2.55}$$

then $(f_n)_{n=1}^{\infty}$ is a nearly strongly mixing sequence.

Proof. Equation (2.55) implies equation (2.52) for any sequence $(N_q)_{q=1}^{\infty}$, and all bounded sequences are contained in $SA(H)$. □

Theorem 2.2.16. *Let H be a Hilbert space and let $(f_n)_{n=1}^{\infty} \in SA(H)$. $(f_n)_{n=1}^{\infty}$ is a nearly orthogonal sequence if and only if for all permissible triples of the form $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ and all $h \in \mathbb{N}$, we have*

$$\lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} \langle f_{n+h}, f_n\rangle = 0. \tag{2.56}$$

Proof. For the first direction, let us assume that $(f_n)_{n=1}^{\infty}$ is a nearly orthogonal sequence, let $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ be a permissible triple, and let $H' = H'((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Letting $M = ||(f_n)_{n=1}^{\infty}||_{H'}$ we may apply the definition of nearly orthogonal sequences to see that

$$M^2 = M \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} ||f_n||^2 \geq \sum_{h=0}^{\infty} \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} |\langle f_{n+h}, f_n\rangle|^2 \tag{2.57}$$

$$= \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} ||f_n||^4 + \sum_{h=1}^{\infty} \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} |\langle f_{n+h}, f_n\rangle|^2$$

$$\geq M^2 + \sum_{h=1}^{\infty} \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} |\langle f_{n+h}, f_n\rangle|^2, \text{ so}$$

$$\sum_{h=1}^{\infty} \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} |\langle f_{n+h}, f_n\rangle|^2 = 0, \tag{2.58}$$

which yields the desired result.

Let us now proceed to prove the converse. Let $(g_n)_{n=1}^{\infty} \subseteq H$ satisfy

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} ||g_n||^2 < \infty, \tag{2.59}$$

and let $(N_q)_{q=1}^{\infty}$ be such that

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, g_n\rangle \tag{2.60}$$

exists for every $h \in \mathbb{N}$. By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, so may define $\mathcal{H} = \mathcal{H}((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Let $S : \mathcal{H} \to \mathcal{H}$ denote the unitary operator induced by the left shift. From equation (2.56) we see that

$$\langle S^h((f_n)_{n=1}^{\infty}), (f_n)_{n=1}^{\infty}\rangle_{\mathcal{H}} = 0 \tag{2.61}$$

for every $h \in \mathbb{N}$, so $\{(f_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}$ is a bounded orthogonal set of vectors in $\mathcal{H}$, which yields the desired result. □

Corollary 2.2.17. *Let H be a Hilbert space and let $(f_n)_{n=1}^{\infty} \subseteq H$ be a bounded sequence. $(f_n)_{n=1}^{\infty}$ is a nearly orthogonal sequence if and only if for all $h \in \mathbb{N}$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle f_{n+h}, f_n\rangle = 0. \tag{2.62}$$

Proof. It is clear that equation (2.62) implies equation (2.56) which proves the forward direction of the corollary. We will now prove the reverse direction of the corollary. Let $h \in \mathbb{N}$ and $(M_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ be such that

$$\lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q}\langle f_{n+h}, f_n\rangle \text{ exists.} \tag{2.63}$$

By passing to a subsequence $(N_q)_{q=1}^{\infty}$ of $(M_q)_{q=1}^{\infty}$ we may assume without loss of generality that $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple. Since $(f_n)_{n=1}^{\infty}$ is a nearly orthogonal sequence, we see by Theorem 2.2.16 that

$$0 = \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, f_n\rangle = \lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q}\langle f_{n+h}, f_n\rangle. \tag{2.64}$$

□

Remark 2.2.18. Theorems 2.2.10, 2.2.12, 2.2.14, and other variations of van der Corput's Difference Theorem can all be proven simultaneously by using the main results of [Tse16] after $\mathcal{H}$ has been constructed. However, the main results of [Tse16] cannot be used to prove Theorems 2.2.8 and 2.2.16.

Theorem 2.2.19. *Let $U : H \to H$ be a unitary operator and $f \quad H$.*

(i) $(U^n f)_{n=1}^{\infty}$ *is a completely ergodic sequence if and only if f is an ergodic element of H.*

(ii) $(U^n f)_{n=1}^{\infty}$ *is a nearly weakly mixing sequence if and only if f is a weakly mixing element of H.*

(iii) $(U^n f)_{n=1}^{\infty}$ *is a nearly mildly mixing sequence if and only if f is a mildly mixing element of H.*

(iv) $(U^n f)_{n=1}^{\infty}$ *is a nearly strongly mixing sequence if and only if f is a strongly mixing element of H.*

(v) $(U^n f)_{n=1}^{\infty}$ *is a nearly orthogonal sequence if and only if $\{U^n f\}_{n=1}^{\infty}$ is a set of orthogonal vectors.*

Proof of i. Firstly, we recall that the Mean Ergodic Theorem tells us that

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}U^h f = Pf, \tag{2.65}$$

where $P : H \to H$ is the projection onto the subspace of U-invariant elements. It follows that f is an ergodic element of (H, U) if and only if Definition 2.2.4(i) holds for $g = f$. We now see that for all permissible triples of the form $((U^n f)_{n=1}^{\infty}, (U^n f)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} U^{n+h}f, U^n f \;=\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} U^h f, f \tag{2.66}$$

$$=\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H} U^h f, f \;. \tag{2.67}$$

It follows from Theorem 2.2.8 that f is a weakly mixing element of (H, U) if and only if $(U^n f)_{n=1}^{\infty}$ is a completely ergodic sequence. □

Proof of ii. We see that for all permissible triples of the form $((U^n f)_{n=1}^{\infty}, (U^n f)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q} U^{n+h}f, U^n f\;\right| = \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q} U^h f, f\;\right| \tag{2.68}$$

$$= \lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} | \langle U^h f, f \rangle |. \tag{2.69}$$

It follows from Lemma 3 of [Ŕ58] and Theorem 2.2.10 that f is a weakly mixing element of $(\mathcal{H}, U)$ if and only if $(U^n f)_{n=1}^{\infty}$ is a nearly weakly mixing sequence. □

Proof of iii. We see that for all permissible triple of the form $((U^n f)_{n=1}^{\infty}, (U^n f)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have

$$\text{IP} - \lim_{h \to \infty} \lim_{q \to \infty} \left| \frac{1}{N_q} \sum_{n=1}^{N_q} \langle U^{n+h} f, U^n f \rangle \right| = \text{IP} - \lim_{h \to \infty} \lim_{q \to \infty} \left| \frac{1}{N_q} \sum_{n=1}^{N_q} \langle U^h f, f \rangle \right| \tag{2.70}$$

$$= \text{IP} - \lim_{h \to \infty} | \langle U^h f, f \rangle |, \tag{2.71}$$

so by Theorem 2.2.12 and Lemma 9.24 of [Fur81], f is a mildly mixing element of $(\mathcal{H}, U)$ if and only if $(U^n f)_{n=1}^{\infty}$ is a nearly mildly mixing sequence. □

Proof of iv. We see that for all permissible triple of the form $((U^n f)_{n=1}^{\infty}, (U^n f)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have

$$\lim_{h \to \infty} \lim_{q \to \infty} \left| \frac{1}{N_q} \sum_{n=1}^{N_q} \langle U^{n+h} f, U^n f \rangle \right| = \lim_{h \to \infty} \lim_{q \to \infty} \left| \frac{1}{N_q} \sum_{n=1}^{N_q} \langle U^h f, f \rangle \right| \tag{2.72}$$

$$= \lim_{h \to \infty} | \langle U^h f, f \rangle |. \tag{2.73}$$

so by Lemma 1 of [Ŕ58] and Theorem 2.2.14, f is a strongly mixing element of $(\mathcal{H}, U)$ if and only if $(U^n f)_{n=1}^{\infty}$ is a nearly strongly mixing sequence. □

Proof of v. We see that for all permissible triple of the form $((U^n f)_{n=1}^{\infty}, (U^n f)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have

$$\sum_{h=1}^{\infty} \left| \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle U^{n+h} f, U^n f \rangle \right|^2 = \sum_{h=1}^{\infty} \left| \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle U^h f, f \rangle \right|^2 = \sum_{h=1}^{\infty} | \langle U^h f, f \rangle |^2. \tag{2.74}$$

We now see from Theorem 2.2.16 that $(U^n f)_{n=1}^{\infty}$ is a nearly orthogonal sequence if and only if $\{U^n f\}_{n=1}^{\infty}$ is an orthogonal set of vectors. □

The following lemma shows us why we have had to work with subsequences $(N_q)_{q=1}^{\infty}$ when taking limits instead of just using limit supremums.

Lemma 2.2.20. *If $\{f_n\}_{n=1}^{\infty}$ H is any sequence for which*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} ||f_n|| < \infty, \tag{2.75}$$

then there exists $(g_n)_{n=1}^{\infty}$ $UB(H)$ for which

$$\limsup_{N\to\infty} |\frac{1}{N}\sum_{n=1}^{N} \ f_{n+h}, g_n \ | = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} ||f_n||, \tag{2.76}$$

for every h N $\{0\}$.

Proof. Let $(N_k)_{k=1}^{\infty}$ be such that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} ||f_n|| = \lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} ||f_n||. \tag{2.77}$$

By passing to a subsequence, we may assume without loss of generality that

$$\frac{1}{N_{k+1}}\sum_{n=1}^{N_k} ||f_n|| < \frac{1}{k}. \tag{2.78}$$

Let $\rho : \mathbb{N}^2 \to \mathbb{N}$ be any bijection. For $N_{\rho(m,h)} < n \leq N_{\rho(m,h)+1}$, let $g_n = \frac{f_{n+h}}{||f_{n+h}||}$. Now let h N be arbitrary, and note that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} ||f_n|| \geq \limsup_{N\to\infty} \left|\frac{1}{N}\sum_{n=1}^{N} \ f_{n+h}, g_n \ \right| \tag{2.79}$$

$$\geq \overline{\lim_{m\to\infty}} \left|\frac{1}{N_{\rho(m,h)+1}} \sum_{n=1}^{N_{\rho(m,h)+1}} \ f_{n+h}, g_n \ \right|$$

$$= \overline{\lim_{m\to\infty}} \frac{1}{N_{\rho(m,h)+1}} \left| \sum_{n=1}^{N_{\rho(m,h)}} \ f_{n+h}, g_n \ + \sum_{n=N_{\rho(m,h)}+1}^{N_{\rho(m,h)+1}} ||f_{n+h}|| \right|$$

$$\geq \overline{\lim_{m\to\infty}} \frac{1}{N_{\rho(m,h)+1}} \left(\sum_{n=N_{\rho(m,h)}+1}^{N_{\rho(m,h)+1}} ||f_{n+h}|| - \left| \sum_{n=1}^{N_{\rho(m,h)}} \ f_{n+h}, g_n \ \right| \right)$$

$$\geq \overline{\lim_{m\to\infty}} \frac{1}{N_{\rho(m,h)+1}} \sum_{n=1}^{N_{\rho(m,h)+1}} ||f_{n+h}|| - \frac{2}{\rho(m,h)} = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} ||f_n||.$$

□

2.3 Properties of Completely Ergodic, Nearly Weakly Mixing, Nearly Mildly Mixing, and Nearly Strongly Mixing Sequences

In this section we will demonstrate the difference between Theorems 2.1.2, 2.2.8, 2.2.10, and 2.2.12 by examining properties of completely ergodic, nearly weakly mixing, and nearly mildly mixing sequences. To this end, we will also introduce the notions of invariant, compact, and rigid sequences (cf. Definition 2.3.2) motivated by the corresponding notions in ergodic theory.

Lemma 2.3.1. *Let H be a Hilbert space.*

(i) *Let $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \quad SA(H)$ and let*

$$(x_n)_{n=1}^{\infty} \quad \text{Span}_{\mathbb{C}}(\{(f_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty} \quad \{(g_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}). \tag{2.80}$$

- (a) *If $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are completely ergodic sequences, then $(x_n)_{n=1}^{\infty}$ is also a completely ergodic sequence.*
- (b) *If $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are nearly weakly mixing sequences, then $(x_n)_{n=1}^{\infty}$ is also a nearly weakly mixing sequence.*
- (c) *If $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are nearly mildly mixing sequences, then $(x_n)_{n=1}^{\infty}$ is also a nearly mildly mixing sequence.*
- (d) *If $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are nearly strongly mixing sequences, then $(x_n)_{n=1}^{\infty}$ is also a nearly strongly mixing sequence.*

(ii) *Let $(f_n)_{n=1}^{\infty} \quad SA(H)$ and let $\{(g_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty} \quad SA(H)$ be a family of sequences for which*

$$\lim_{m\to\infty}\left(\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||f_n - g_{n,m}||^2\right) = 0. \tag{2.81}$$

- (a) *If $(g_{n,m})_{n=1}^{\infty}$ is a completely ergodic sequence for all $m \quad \mathbb{N}$, then $(f_n)_{n=1}^{\infty}$ is also a completely ergodic sequence.*
- (b) *If $(g_{n,m})_{n=1}^{\infty}$ is a nearly weakly mixing sequence for all $m \quad \mathbb{N}$, then $(f_n)_{n=1}^{\infty}$ is also a nearly weakly mixing sequence.*
- (c) *If $(g_{n,m})_{n=1}^{\infty}$ is a nearly mildly mixing sequence for all $m \quad \mathbb{N}$, then $(f_n)_{n=1}^{\infty}$ is also a nearly mildly mixing sequence.*
- (d) *If $(g_{n,m})_{n=1}^{\infty}$ is a nearly strongly mixing sequence for all $m \quad \mathbb{N}$, then $(f_n)_{n=1}^{\infty}$ is also a nearly strongly mixing sequence.*

(iii) If $(f_n)_{n=1}^{\infty} \quad UB(H)$ *is a completely ergodic sequence, then*

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} f_n|| = 0. \tag{2.82}$$

(iv) If $(f_n)_{n=1}^{\infty} \quad SA(H)$ *is a completely ergodic sequence, then*

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} f_n|| = 0. \tag{2.83}$$

Items (i) and (ii) can be interpreted as statements that the set of ergodic, weak mixing, mild mixing, and strong mixing elements of $H \; = H \; ((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ are each closed linear subspaces of H . The proofs of all items of (i) and (ii) are routine and therefore omitted. While it is clear that (iv) implies (iii), we still give a separate proof of (iii) since it is similar in spirit to the classical proofs of any of Theorems 2.1.2(i)-2.1.2(iii). A proof of (iv) can be obtained in a similar fashion, but we use the machinery of Section 2.2 to provide an alternative perspective. Yet another proof of (iv) can be obtained by using the methodology of the proof of Lemma 3.2.6.

Proof of (iii). Let $(f_n)_{n=1}^{\infty} \quad H$ be a completely ergodic sequence that is uniformly bounded in norm by 1. Let $(M_q)_{q=1}^{\infty} \quad \mathbb{N}$ be any sequence for which

$$\lim_{q\to\infty} ||\frac{1}{M_q}\sum_{n=1}^{M_q} f_n|| \tag{2.84}$$

exists, and let $(N_q)_{q=1}^{\infty}$ be a subsequence of $(M_q)_{q=1}^{\infty}$ for which $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple. Let $\epsilon > 0$ be arbitrary and let $K \geq \frac{1}{\epsilon}$ be such that

$$\left| \frac{1}{N_q H}\sum_{h=1}^{H}\sum_{n=1}^{N_q} \; f_{n+h}, f_n \; \right| < \epsilon, \tag{2.85}$$

whenever $H > K$ and $N_q > N(H)$. Letting $H = K^2$, we see that

$$\lim_{q\to\infty} ||\frac{1}{M_q}\sum_{n=1}^{M_q} f_n||^2 = \lim_{q\to\infty} ||\frac{1}{N_q}\sum_{n=1}^{N_q} f_n||^2 = \lim_{q\to\infty} ||\frac{1}{H}\sum_{h=1}^{H}\frac{1}{N_q}\sum_{n=1}^{N_q} f_{n+h}||^2 \tag{2.86}$$

$$\leq \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} ||\frac{1}{H}\sum_{h=1}^{H} f_{n+h}||^2 = \lim_{q\to\infty} \frac{1}{N_q H^2}\sum_{n=1}^{N_q}\sum_{1\leq h_1,h_2\leq H} \quad f_{n+h_1}, f_{n+h_2}$$

$$
\begin{aligned}
&= \frac{1}{H} + \lim_{q\to\infty} \sum_{h=1}^{H} \frac{2(H-h)}{N_q H^2} \mathrm{Re}(\sum_{n=1}^{N_q} f_{n+h}, f_n \;) && (2.87)\\
&\leq \frac{1}{H} + \sum_{t=1}^{H-1} \frac{2}{H} \lim_{q\to\infty} \left| \sum_{h=1}^{t} \frac{1}{N_q H} \sum_{n=1}^{N_q} f_{n+h}, f_n \;\right| \\
&\leq \frac{2K+1}{H} + \sum_{t=K+1}^{H-1} \frac{2}{H} \lim_{q\to\infty} \left| \sum_{h=1}^{t} \frac{1}{N_q H} \sum_{n=1}^{N_q} f_{n+h}, f_n \;\right| \\
&\leq \frac{2K+1}{H} + \frac{2(H-K-1)\epsilon}{H} \leq \frac{3K}{K^2} + 2\epsilon \leq 5\epsilon.
\end{aligned}
$$

□

Proof of (iv). Let $(M_q)_{q=1}^{\infty}$ N be any sequence for which

$$\lim_{q\to\infty} \|\frac{1}{M_q} \sum_{n=1}^{M_q} f_n\| \tag{2.88}$$

exists, and let $(N_q)_{q=1}^{\infty}$ be a subsequence of $(M_q)_{q=1}^{\infty}$ for which $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple. Let $\mathcal{H} = \mathcal{H}((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, let $S : \mathcal{H} \to \mathcal{H}$ be the unitary operator induced by the left shift, let $\mathcal{H}_I := \{\xi \quad \mathcal{H} \mid S\xi = \xi\}$, and let $P : \mathcal{H} \to \mathcal{H}_I$ be the orthogonal projection. Since $(f_n)_{n=1}^{\infty}$ is a completely ergodic sequence we see that it is an ergodic element of $(\mathcal{H}, S)$, so $P(f_n)_{n=1}^{\infty} = (0)_{n=1}^{\infty}$. By the Mean Ergodic Theorem we see that

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} S^h (f_n)_{n=1}^{\infty} = P(f_n)_{n=1}^{\infty} = (0)_{n=1}^{\infty}. \tag{2.89}$$

We would like to perform a calculation similar to that in the first line of (2.86), but since we are now working with unbounded sequences we need to be more careful. More specifically, we would like to show that for any H N we have

$$\lim_{q\to\infty} \|\frac{1}{N_q} \sum_{n=1}^{N_q} f_n\|^2 = \lim_{q\to\infty} \|\frac{1}{H} \sum_{h=1}^{H} \frac{1}{N_q} \sum_{n=1}^{N_q - H} f_{n+h}\|^2. \tag{2.90}$$

To this end, let us assume for the sake of contradiction that for some H N, some $\epsilon > 0$, and some subsequence $(N_q)_{q=1}^{\infty}$ of $(N_q)_{q=1}^{\infty}$ we have

$$\epsilon \leq \frac{1}{HN_q} \sum_{h=1}^{H} \|(H-h+1) f_{N_{q-H+h}}\| \leq \frac{1}{N_q} \sum_{h=1}^{H} \|f_{N_{q-H+h}}\|, \tag{2.91}$$

for all q N. It follows that

$$||(f_n)_{n=1}^{\infty}||_H = \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} ||f_n||^2 \geq \lim_{q\to\infty} \frac{1}{N_q}\sum_{h=1}^{H} ||f_{N_{q-H+h}}||^2 \tag{2.92}$$
$$\geq \lim_{q\to\infty} \frac{1}{HN_q}(\sum_{h=1}^{H} ||f_{N_{q-H+h}}||)^2 \geq \lim_{q\to\infty} \frac{1}{HN_q}(N_q\epsilon)^2 = \infty,$$

which yields the desired contradiction and shows that equation (2.90) is true. Now let $\epsilon > 0$ be arbitrary and observe that for sufficiently large H N we have

$$\epsilon \geq ||\frac{1}{H}\sum_{h=1}^{H} S^h(f_n)_{n=1}^{\infty}||_H^2 = ||(\frac{1}{H}\sum_{h=1}^{H} f_{n+h})_{n=1}^{\infty}||_H^2 = \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||\frac{1}{H}\sum_{h=1}^{H} f_{n+h}||^2$$
$$\geq \left(\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||\frac{1}{H}\sum_{h=1}^{H} f_{n+h}||\right)^2 \geq \left(\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q-H}||\frac{1}{H}\sum_{h=1}^{H} f_{n+h}||\right)^2 \tag{2.93}$$
$$\geq \left(\lim_{q\to\infty}||\frac{1}{N_q}\sum_{n=1}^{N_q-H}\frac{1}{H}\sum_{h=1}^{H} f_{n+h}||\right)^2 = \left(\lim_{q\to\infty}||\frac{1}{N_q}\sum_{n=1}^{N_q} f_{n+h}||\right)^2$$
$$= \left(\lim_{q\to\infty}||\frac{1}{M_q}\sum_{n=1}^{M_q} f_{n+h}||\right)^2.$$

□

Definition 2.3.2. *Let H be a Hilbert space, $U : H \to H$ be a unitary operator, and f H.*

*(i) f is an **invariant element of** (H, U) if $Uf = f$.*

*(ii) f is a **compact element of** (H, U) if $(U^n f)_{n=1}^{\infty}$ is precompact in the norm topology of H. Equivalently, f is a compact element if for all $\epsilon > 0$, there exists K N for which*

$$\sup_{m \quad \mathbb{N}} \min_{1\leq k\leq K} ||U^m f - U^k f|| < \epsilon. \tag{2.94}$$

*(iii) f is a **rigid element of** (H, U) if f is in the norm closure of $(U^n f)_{n=1}^{\infty}$. Equivalently, f is a rigid element if there exists $(k_m)_{m=1}^{\infty}$ N for which*

$$\lim_{m\to\infty} ||U^{k_m} f - f|| = 0. \tag{2.95}$$

If (H, U) is understood from the context, then we may say that f is an invariant, compact, or rigid element.

Definition 2.3.3. *Let H be a Hilbert space and $(c_n)_{n=1}^{\infty} \subseteq SA(H)$.*

(i) $(c_n)_{n=1}^{\infty}$ *is an* ***invariant sequence*** *if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||c_{n+1}-c_n||^2=0. \tag{2.96}$$

(ii) $(c_n)_{n=1}^{\infty}$ *is a* ***compact sequence***[2] *if for all $\epsilon>0$ and all permissible triples $((c_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, there exists $K \in \mathbb{N}$ for which*

$$\sup_{m\in\mathbb{N}}\min_{1\le k\le K}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||c_{n+m}-c_{n+k}||^2<\epsilon. \tag{2.97}$$

(iii) $(c_n)_{n=1}^{\infty} \subseteq H$ *is a* ***rigid sequence*** *if for all permissible triples $((c_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ there exists $(k_m)_{m=1}^{\infty} \subseteq \mathbb{N}$ for which*

$$\lim_{m\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||c_{n+k_m}-c_n||^2=0. \tag{2.98}$$

Remark 2.3.4. We see that $(f_n)_{n=1}^{\infty} \subseteq SA(H)$ is an invariant (compact, rigid) sequence if and only if for all permissible triple of the form $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, $(f_n)_{n=1}^{\infty}$ is an invariant (compact, rigid) element of $(H((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}), (f_n)_{n=1}^{\infty}, S)$. Furthermore, we see that if $U : H \to H$ is a unitary operator and $f \in H$, then $(U^n f)_{n=1}^{\infty}$ is an invariant (compact, rigid) sequence if and only if f is an invariant (compact, rigid) element of (H, U). It is also worth noting for the reader who reads Section 2.6 that the definitions appearing in Definition 2.3.3 could be named “nearly invariant sequences", “nearly compact sequences", and “nearly rigid sequences" since the definitions use Cesàro averages. We choose to simplify the names and omit the word ’nearly’ since the notions of invariant, compact, and rigid sequences of vectors in a Hilbert space are not currently being used elsewhere.

We will now recall the classical compact-weak mixing decomposition of a Hilbert space for use in some of the upcoming proofs. An initial form of this result traces back to the work of B. Koopman and J. von Neumann [KN32] (see also Theorem 2.3 in [Ber96]) while the result in its full generality was obtained by K. Jacobs [Jac56], K. de Leeuw, and I Glicksberg [LG61] (see also Chapter 2.4 in [Kre85] and Example 16.25 in [Eis+15]).

[2] This definition was motivated by Definition 3.13 in [MRR19].

Theorem 2.3.5 (Jacobs-de Leeuw-Glicksberg Decomposition)**.** *Given a Hilbert space H and a unitary operator $U : H \to H$ let*

$$\begin{aligned} H_c &:= \{\xi \quad H \mid \xi \text{ is a compact element of } H\} \text{ and} \\ H_{wm} &:= \{\xi \quad H \mid \xi \text{ is a weakly mixing element of } H\}. \end{aligned} \tag{2.99}$$

We have that $H = H_c \quad H_{wm}$.

Theorem 2.3.6. *Let H be a Hilbert space and suppose that either $(x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \quad SA(H)$ or $(x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \quad UB(H)$.*

(i) $(x_n)_{n=1}^{\infty}$ is a completely ergodic sequence if and only if for all invariant sequences $(c_n)_{n=1}^{\infty}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \quad x_n, c_n \quad = 0. \tag{2.100}$$

(ii) $(x_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence if and only if for all compact sequence $(c_n)_{n=1}^{\infty}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \quad x_n, c_n \quad = 0. \tag{2.101}$$

(iii) If $(x_n)_{n=1}^{\infty}$ is a nearly mildly mixing sequence if and only if for all rigid sequences $(c_n)_{n=1}^{\infty}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \quad x_n, c_n \quad = 0. \tag{2.102}$$

We begin by observing that for all 3 statements the forwards direction for $(x_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty} \quad SA(H)$ implies the forwards direction for $(x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \quad UB(H)$, but the backwards directions need to be addressed separately.

Proof of (i)*.* For the first direction let us assume that $(x_n)_{n=1}^{\infty} \quad SA(H)$ is a completely ergodic sequence and $(c_n)_{n=1}^{\infty} \quad SA(H)$ is invariant. Let $(M_q)_{q=1}^{\infty} \quad \mathbb{N}$ be such that

$$\lim_{q\to\infty} \frac{1}{M_q}\sum_{n=1}^{M_q} \quad x_n, c_n \tag{2.103}$$

exists. Let $(N_q)_{q=1}^{\infty}$ be any subsequence of $(M_q)_{q=1}^{\infty}$ for which $((x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, and let $\mathcal{H} = \mathcal{H}((x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Since $(x_n)_{n=1}^{\infty}$ is a completely ergodic sequence and $(c_n)_{n=1}^{\infty}$ is an invariant sequence, we see that

$$
\begin{aligned}
0 &= \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} x_{n+h}, c_n \\
&= \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} x_n, c_{n-h} \\
&= \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} x_n, c_n \ = \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} x_n, c_n \\
&= \lim_{q\to\infty} \frac{1}{M_q}\sum_{n=1}^{M_q} x_n, c_n \ .
\end{aligned}
$$

For the next direction, let us assume that $(x_n)_{n=1}^{\infty}$ satisfies equation (2.100) whenever $(c_n)_{n=1}^{\infty}$ is an invariant sequence. Let $(N_q)_{q=1}^{\infty}$ be such that $((x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple and let $\mathcal{H} = \mathcal{H}((x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Let $S : \mathcal{H} \to \mathcal{H}$ denote the left shift unitary operator, and by the Mean Ergodic Theorem we see that

$$
(c_n)_{n=1}^{\infty} := \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} S^h (x_n)_{n=1}^{\infty} \tag{2.104}
$$

is the projection of $(x_n)_{n=1}^{\infty}$ onto the subspace of elements of $\mathcal{H}$ that are invariant under S. If follows that

$$
0 = \ (c_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty} \ _{\mathcal{H}} = \ (c_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \ _{\mathcal{H}} = ||(c_n)_{n=1}^{\infty}||_{\mathcal{H}}^2 \ . \tag{2.105}
$$

Since $(N_q)_{q=1}^{\infty}$ was arbitrary, the desired result follows from Theorem 2.2.8.

Lastly, we observe that if $(x_n)_{n=1}^{\infty} \quad B_M$ then $\frac{1}{H}\sum_{h=1}^{H}(x_{n+h})_{n=1}^{\infty} \quad B_M$ for all $H \quad \mathbb{N}$, so by Lemma 2.2.2 we see that $(c_n)_{n=1}^{\infty} \quad B_M$, hence the desired result holds if $(x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}$ $UB(\mathcal{H})$. □

Proof of (ii). For the first direction let us assume that $(x_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence and that $(c_n)_{n=1}^{\infty}$ is compact. Let $(M_q)_{q=1}^{\infty} \quad \mathbb{N}$ be such that

$$
\lim_{q\to\infty} \frac{1}{M_q}\sum_{n=1}^{M_q} x_n, c_n \tag{2.106}
$$

exists. Let $(N_q)_{q=1}^{\infty}$ be any subsequence of $(M_q)_{q=1}^{\infty}$ for which $((x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple and let $\mathcal{H} = \mathcal{H}((x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Since $(x_n)_{n=1}^{\infty}$ is a weakly mixing element of $\mathcal{H}$ and $(c_n)_{n=1}^{\infty}$ is a compact element of $\mathcal{H}$, we see as a consequence of the Jacobs-de Leeuw-Glicksberg Decomposition that

$$0 = \langle (x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \rangle_{\mathcal{H}} \tag{2.107}$$

$$= \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle x_n, c_n \rangle = \lim_{q\to\infty} \frac{1}{M_q} \sum_{n=1}^{M_q} \langle x_n, c_n \rangle . \tag{2.108}$$

Since $(M_q)_{q=1}^{\infty}$ was arbitrary, the desired result follows.

Now let us show that if $(x_n)_{n=1}^{\infty} \in \mathcal{H}$ satisfies equation (2.101) whenever $(c_n)_{n=1}^{\infty} \in \mathcal{H}$ is a compact sequence, then $(x_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence. Let $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ be such that $((x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple. Let $\mathcal{H} = \mathcal{H}((x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ and let $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{wm}$ be the Jacobs-de Leeuw-Glicksberg Decomposition of $\mathcal{H}$ with respect to the shift map S. Let $(x_{n,c})_{n=1}^{\infty} \in \mathcal{H}_c$ and $(x_{n,wm})_{n=1}^{\infty} \in \mathcal{H}_{wm}$ be such that $(x_n)_{n=1}^{\infty} = (x_{n,c})_{n=1}^{\infty} + (x_{n,wm})_{n=1}^{\infty}$. We see that

$$0 = \langle (x_{n,c})_{n=1}^{\infty}, (x_n)_{n=1}^{\infty} \rangle_{\mathcal{H}} = \langle (x_{n,c})_{n=1}^{\infty}, (x_{n,c})_{n=1}^{\infty} + (x_{n,wm})_{n=1}^{\infty} \rangle_{\mathcal{H}} \tag{2.109}$$

$$= \langle (x_{n,c})_{n=1}^{\infty}, (x_{n,c})_{n=1}^{\infty} \rangle_{\mathcal{H}} = ||(x_{n,c})_{n=1}^{\infty}||_{\mathcal{H}}^2 . \tag{2.110}$$

It follows that $(x_n)_{n=1}^{\infty} = (x_{n,wm})_{n=1}^{\infty}$.

It only remains to show that the desired result holds when $(x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \in UB(\mathcal{H})$. To this end, we first recall that $(c_n)_{n=1}^{\infty}$ is in the weak closure of $\{(x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}$ (cf. Theorem 2.25 in [BM10]). We now see that if $(x_n)_{n=1}^{\infty} \in B_M$ then $(x_{n+h})_{n=1}^{\infty} \in B_M$ for all $h \in \mathbb{N}$, so by Lemma 2.2.2 we see that $(c_n)_{n=1}^{\infty} \in B_M \subseteq UB(\mathcal{H})$. □

Proof of (iii). For the first direction let us assume that $(x_n)_{n=1}^{\infty}$ is a nearly mildly mixing sequence and $(c_n)_{n=1}^{\infty}$ is rigid. Since $(c_n)_{n=1}^{\infty}$ is rigid, let $(k_m)_{m=1}^{\infty}$ be such that

$$\lim_{m\to\infty} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} ||c_{n+k_m} - c_n||^2 = 0. \tag{2.111}$$

Now let $(M_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ be any sequence for which

$$\lim_{q\to\infty} \frac{1}{M_q} \sum_{n=1}^{M_q} \; x_n, c_n \tag{2.112}$$

exists, and let $(N_q)_{q=1}^{\infty}$ be any subsequence of $(M_q)_{q=1}^{\infty}$ for which

$$\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_{n+h}, c_n \tag{2.113}$$

exists for every h N. We see that

$$0 = \lim_{h\to\infty} \left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_{n+h}, c_n \; \right| = \lim_{m\to\infty} \left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_{n+k_m}, c_n \; \right| \tag{2.114}$$

$$\geq \lim_{m\to\infty} \left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_{n+k_m}, c_{n+k_m} \; \right| - \lim_{m\to\infty} \left| \overline{\lim_{q\to\infty}} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_{n+k_m}, c_{n+k_m} - c_n \; \right| \tag{2.115}$$

$$\geq \left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_n, c_n \; \right| - \lim_{m\to\infty} \left((\overline{\lim_{q\to\infty}} \frac{1}{N_q} \sum_{n=1}^{N_q} \|x_{n+k_m}\|^2)^{\frac{1}{2}} (\overline{\lim_{q\to\infty}} \frac{1}{N_q} \sum_{n=1}^{N_q} \|c_{n+k_m} - c_n\|^2)^{\frac{1}{2}} \right) \tag{2.116}$$

$$= \left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_n, c_n \; \right| = \left| \lim_{q\to\infty} \frac{1}{M_q} \sum_{n=1}^{M_q} \; x_n, c_n \; \right|. \tag{2.117}$$

Since $\{M_q\}_{q=1}^{\infty}$ was arbitrary, the desired result follows.

For the next direction let us assume that $(x_n)_{n=1}^{\infty}$ is not a nearly mildly mixing sequence. We will use some knowledge about idempotent ultrafilters that is discussed in Section 2.7. Let $(N_q)_{q=1}^{\infty}$ N be such that

$$\text{IP} - \lim_{h\to\infty} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_{n+h}, x_n \; = 0. \tag{2.118}$$

Let $\epsilon > 0$ and $(v_i)_{i=1}^{\infty}$ N be such that

$$\mathrm{FS}(v_i)_{i=1}^{\infty} \quad \{h \quad \mathbb{N} \mid \left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_{n+h}, x_n \; \right| \geq \epsilon\}. \tag{2.119}$$

Let $p \quad \beta\mathbb{N}$ be an idempotent ultrafilter for which $\mathrm{FS}(v_i)_{i=1}^{\infty} \quad p$. Let $\mathcal{H} = \mathcal{H}\,((x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Since $B := \{\vec{x} \quad \mathcal{H} \mid \epsilon \leq ||\vec{x}||_{\mathcal{H}} \leq ||(x_n)_{n=1}^{\infty}||_{\mathcal{H}}\}$ is compact in the weak topology, let $\tilde{x} \quad B$ be such that

$$\tilde{x} = p - \lim_{h\to\infty} (x_{n+h})_{n=1}^{\infty}. \tag{2.120}$$

Let $S : \mathcal{H} \to \mathcal{H}$ denote the unitary operator induced by the left shift. To see that $\tilde{x}$ is a rigid element of $(\mathcal{H}, S)$ we observe that

$$\begin{aligned} &||\tilde{x} - S^h\tilde{x}||_{\mathcal{H}} = \; \tilde{x} - S^h\tilde{x}, \tilde{x} - S^h\tilde{x} \; _{\mathcal{H}} && (2.121)\\ =& \; \tilde{x}, \tilde{x} \; _{\mathcal{H}} - \; \tilde{x}, S^h\tilde{x} \; _{\mathcal{H}} - \; S^h\tilde{x}, \tilde{x} \; _{\mathcal{H}} + \; S^h\tilde{x}, S^h\tilde{x} \; _{\mathcal{H}}\\ =& 2||\tilde{x}||_{\mathcal{H}} - 2\mathrm{Re}(\; S^h\tilde{x}, \tilde{x} \; _{\mathcal{H}}), \text{ hence}\\ &p - \lim_{h\to\infty} = ||\tilde{x} - S^h\tilde{x}||_{\mathcal{H}} = p - \lim_{h\to\infty}\left(2||\tilde{x}||_{\mathcal{H}} - 2\mathrm{Re}(\; S^h\tilde{x}, \tilde{x} \; _{\mathcal{H}})\right)\\ =& 2||\tilde{x}||_{\mathcal{H}} - 2\mathrm{Re}(\; p - \lim_{h\to\infty} S^h\tilde{x}, \tilde{x} \; _{\mathcal{H}}) = 2||\tilde{x}||_{\mathcal{H}} - 2\mathrm{Re}(\; \tilde{x}, \tilde{x} \; _{\mathcal{H}}) = 0. \end{aligned}$$

It now suffices to take $(c_n)_{n=1}^{\infty} = \tilde{x}$ and observe that

$$\begin{aligned} &\left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \; x_n, c_n \; \right| = | \; (x_n)_{n=1}^{\infty}, \tilde{x} \; _{\mathcal{H}} | = \left| \; (x_n)_{n=1}^{\infty}, p - \lim_{h\to\infty} S^h(x)_{n=1}^{\infty} \; _{\mathcal{H}} \right| && (2.122)\\ =& p - \lim_{h\to\infty} | \; (x_n)_{n=1}^{\infty}, (x_{n+h})_{n=1}^{\infty} \; _{\mathcal{H}} | \geq 0. \end{aligned}$$

Lastly, we observe that if $(x_n)_{n=1}^{\infty} \quad B_M$ then $(x_{n+h})_{n=1}^{\infty} \quad B_M$ for all $h \quad \mathbb{N}$, so by Lemma 2.2.2 we see that $(c_n)_{n=1}^{\infty} \quad B_M$, hence the desired result holds if $(x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}$ $UB(\mathcal{H})$. □

Remark 2.3.7. To see why we require that $(x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \quad SA(\mathcal{H})$ or $(x_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}$ $UB(\mathcal{H})$, let us consider the sequence $(x_n)_{n=1}^{\infty} \quad SA(\mathbb{C})$ given by

$$x_n = \begin{cases} m & \text{if } m^4 - m \leq n < m^4 + m \\ 0 & \text{else} \end{cases}. \tag{2.123}$$

We see that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |x_n|^2 = \lim_{M\to\infty} \frac{1}{M^4+M} \sum_{m=1}^{M} 2m \cdot m^2 = \frac{2}{3} < \infty. \tag{2.124}$$

It is also clear that $(x_n)_{n=1}^\infty$ is an invariant sequence by construction, so it cannot be a completely ergodic sequence. However, for any $(c_n)_{n=1}^\infty \quad UB(\mathbb{C})$, there is a representative $(c_n)_{n=1}^\infty$ in the same equivalence class as $(c_n)_{n=1}^\infty$ for which $c_n = 0$ whenever $m^4 - m \le n \le m^4 + m$, so

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \quad x_n, c_n \quad = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \quad x_n, c_n \quad = 0. \tag{2.125}$$

Lemma 2.3.8. *Let H_1, H_2, and H_3 be subsets of Hilbert spaces*[3] *and let $\cdot : H_1 \times H_2 \to H_3$ be a bilinear map for which $||f_1 \cdot f_2||_{H_3} \le C||f_1||_{H_1}||f_2||_{H_2}$ for some $C > 0$ and any $(f_1, f_2) \quad H_1 \times H_2$. Let $(f_{n,1})_{n=1}^\infty \quad UB(H_1)$ and $(f_{n,2})_{n=1}^\infty \quad UB(H_2)$.*

(i) If $(f_{n,1})_{n=1}^\infty$ and $(f_{n,2})_{n=1}^\infty$ are invariant sequences, then $(f_{n,1} \cdot f_{n,2})_{n=1}^\infty$ is an invariant sequence.

(ii) If $(f_{n,1})_{n=1}^\infty$ and $(f_{n,2})_{n=1}^\infty$ are compact sequences, then $(f_{n,1} \cdot f_{n,2})_{n=1}^\infty$ is a compact sequence.

(iii) If $(f_{n,1})_{n=1}^\infty$ is a compact sequence and $(f_{n,2})_{n=1}^\infty$ is a rigid sequence, then $(f_{n,1} \cdot f_{n,2})_{n=1}^\infty$ is a rigid sequence.

For the proofs of items (i)-(iii) we will assume for the sake of convenience that $(f_{n,1})_{n=1}^\infty$ and $(f_{n,2})_{n=1}^\infty$ are both uniformly bounded in norm by 1.

Proof of (i). We see that

[3] We will be applying this lemma to multiply sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ where $(y_n)_{n=1}^\infty$ is either a bounded sequence of complex numbers or a bounded sequence of vectors in $L^\infty(X,\mu)$. $L^\infty(X,\mu)$ is not a Hilbert space, and $L^2(X,\mu)$ is not closed under multiplication, which is why we have this strange set up.

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||f_{n+1,1}\cdot f_{n+1,2}-f_{n,1}\cdot f_{n,2}||_{H_3}^2 \tag{2.126}$$

$$\leq \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||f_{n+1,1}\cdot f_{n+1,2}-f_{n+1,1}\cdot f_{n,2}||_{H_3}^2 \tag{2.127}$$

$$+\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||f_{n+1,1}\cdot f_{n,2}-f_{n,1}\cdot f_{n,2}||_{H_3}^2$$

$$\leq \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}C^2||f_{n+1,1}||_{H_1}^2||^2 f_{n+1,2}-f_{n,2}||_{H_2}^2 \tag{2.128}$$

$$+\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}C^2||f_{n+1,1}-f_{n,1}||_{H_1}^2||f_{n,2}||_{H_2}^2$$

$$\leq C^2\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||f_{n+1,2}-f_{n,2}||_{H_2}^2+C^2\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||f_{n+1,1}-f_{n,1}||_{H_1}^2=0 \tag{2.129}$$

□

Proof of (ii). For $i \quad \{1,2,3\}$ let $S_i : H_i \to H_i$ denote the left shift unitary operator as in Remark 2.2.3. Let $(M_q)_{q=1}^{\infty} \quad \mathbb{N}$ be such that $((f_{n,1}\cdot f_{n,2})_{n=1}^{\infty},(f_{n,1}\cdot f_{n,2})_{n=1}^{\infty},(M_q)_{q=1}^{\infty})$ is a permissible triple. Let $(N_q)_{q=1}^{\infty}$ be a subsequence of $(M_q)_{q=1}^{\infty}$ for which $((f_{n,1})_{n=1}^{\infty},(f_{n,2})_{n=1}^{\infty}$, $(N_q)_{q=1}^{\infty})$ is also a permissible triple. For $i \quad \{1,2,3\}$ and $(g_n)_{n=1}^{\infty} \quad H_i$ let

$$|||(g_n)_{n=1}^{\infty}|||_i^2=\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||g_n||_{H_i}^2. \tag{2.130}$$

We also note that a sequence $(f_n)_{n=1}^{\infty} \quad H_i$ is compact if and only if $\{S_i^h(f_n)_{n=1}^{\infty}\}_{h=1}^{\infty}$ is precompact in the topology of $UB(H_i)$ induced by the seminorm $|||\cdot|||_i$. When we give $UB(H_1)\times UB(H_2)$ the product topology, we see that

$$\overline{\{(S_1\times S_2)^h(f_{n,1},f_{n,2})_{n=1}^{\infty}\}_{h=1}^{\infty}} \quad \overline{\{(S_1^{h_1}\times S_2^{h_2})(f_{n,1},f_{n,2})_{n=1}^{\infty}\}_{(h_1,h_2) \quad \mathbb{N}^2}} \tag{2.131}$$
$$=\overline{\{S_1^h(f_{n,1})_{n=1}^{\infty}\}_{h=1}^{\infty}}\times\overline{\{S_2^h(f_{n,2})_{n=1}^{\infty}\}_{h=1}^{\infty}},$$

so $\{(S_1\times S_2)^h(f_{n,1},f_{n,2})_{n=1}^{\infty}\}_{h=1}^{\infty}$ is also precompact in the product topology. Recall that the product topology on $UB(H_1)\times UB(H_2)$ is induced by the seminorm $|||\cdot|||_4$ given by

$$|||(f_{n,1})_{n=1}^{\infty},(f_{n,2})_{n=1}^{\infty}|||_4=|||(f_{n,1})_{n=1}^{\infty}|||_1+|||(f_{n,2})_{n=1}^{\infty}|||_2. \tag{2.132}$$

Since $\{S_3^h(f_{n,1} \cdot f_{n,2})_{n=1}^{\infty}\}_{h=1}^{\infty}$ is the image of $\{(S_1 \times S_2)^h(f_{n,1}, f_{n,2})_{n=1}^{\infty}\}_{h=1}^{\infty}$ under $\cdot$, it suffices to show (with abuse of notation) that $\cdot : UB(H_1) \times UB(H_2) \to UB(H_3)$ is continuous. To this end, let $\epsilon > 0$ be arbitrary and let $(g_{n,1}, g_{n,2})_{n=1}^{\infty}, (e_{n,1}, e_{n,2})_{n=1}^{\infty}$ $UB(H_1) \times UB(H_2)$ be such that $|||(g_{n,1}, g_{n,2})_{n=1}^{\infty} - (e_{n,1}, e_{n,2})_{n=1}^{\infty}|||_4 \le \frac{\epsilon}{CW}$, where

$$W = \max(\sup_{n \quad \mathbb{N}} ||g_{n,1}||, \sup_{n \quad \mathbb{N}} ||e_{n,2}||). \tag{2.133}$$

We now see that

$$\begin{aligned} &|||(g_{n,1} \cdot g_{n,2})_{n=1}^{\infty} - (e_{n,1} \cdot e_{n,2})_{n=1}^{\infty}|||_3 \\ \le &|||(g_{n,1} \cdot g_{n,2})_{n=1}^{\infty} - (g_{n,1} \cdot e_{n,2})_{n=1}^{\infty}|||_3 + |||(g_{n,1} \cdot e_{n,2})_{n=1}^{\infty} - (e_{n,1} \cdot e_{n,2})_{n=1}^{\infty}|||_3 \\ \le &WC|||(g_{n,2})_{n=1}^{\infty} - (e_{n,2})_{n=1}^{\infty}|||_2 + WC|||(g_{n,1})_{n=1}^{\infty} - (e_{n,1})_{n=1}^{\infty}|||_1 \le \epsilon. \end{aligned} \tag{2.134}$$

□

Proof of (iii). The idea behind the proof is to first show that any compact sequence and any rigid sequence are both rigid along a common sequence of integers, then to show that any two rigid sequences that are rigid along a common sequence have a product that is also rigid. We begin by showing that for all $\epsilon > 0$ and all permissible triples $((f_{n,1})_{n=1}^{\infty}, (f_{n,1})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ the set

$$R_1(\epsilon) := \{h \quad \mathbb{N} \mid \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||f_{n+h,1} - f_{n,1}||^2 < \epsilon\} \tag{2.135}$$

is IP . Let $H = H((c_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}$ and let $H = H_c \quad H_{wm}$ be the Jacobs-de Leeuw-Glicksberg Decomposition of H with respect to the unitary operator S induced by the left shift. Since $(f_{n,1})_{n=1}^{\infty}$ is a compact sequence we see that $(f_{n,1})_{n=1}^{\infty} \quad H_c$. Since H_c is also known to be the closure of the span of the eigenfunctions of S, let $\vec{x}_1, \cdots, \vec{x}_K$ be eigenfunctions of S with corresponding eigenvalues of $\lambda_1, \cdots, \lambda_K$ such that

$$||(f_{n,1})_{n=1}^{\infty} - \sum_{i=1}^{K} c_i \vec{x}_i||_H < \frac{\epsilon}{3}, \tag{2.136}$$

with $c_1, \cdots, c_K \quad \mathbb{C}$. Since the set $R_2(\epsilon) := \{h \quad \mathbb{N} \mid ||\vec{x}_i - \lambda_i^h \vec{x}_i||_H < \frac{\epsilon}{3K|c_i|}\}$ is IP (cf. Lemma 2.7.5), and $R_2(\epsilon) \quad R_1(\epsilon)$, we see that $R_1(\epsilon)$ is indeed IP .

Let $((f_{n,1} \cdot f_{n,2})_{n=1}^{\infty}, (f_{n,1} \cdot f_{n,2})_{n=1}^{\infty}, (M_q)_{q=1}^{\infty})$ be a permissible triple and let $(N_q)_{q=1}^{\infty}$ $(M_q)_{q=1}^{\infty}$ be such that $((f_{n,1})_{n=1}^{\infty}, (f_{n,2})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is also a permissible triple so that we may set $H = H((f_{n,1})_{n=1}^{\infty}, (f_{n,2})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. Let $(k_m)_{m=1}^{\infty} \quad \mathbb{N}$ be such that

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||f_{n+k_m,2}-f_{n,2}||^2<\frac{1}{2^m}, \tag{2.137}$$

and observe that we may use the triangle inequality to show that for any k $\mathrm{FS}(k_m)_{m=M}^{\infty}$

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||f_{n+k,2}-f_{n,2}||^2<\frac{1}{2^{M-1}}. \tag{2.138}$$

Since $R_1(\delta)$ is IP for any $\delta>0$, let $(v_i)_{i=1}^{\infty}$ be such that v_M $R_1(2^{-M})\cap\mathrm{FS}(k_m)_{m=M+1}^{\infty}$ for all M $\mathbb{N}$. Letting

$$W=\max(\sup_{n\ \mathbb{N}}||f_{n,1}||,\sup_{n\ \mathbb{N}}||f_{n,2}||), \tag{2.139}$$

we now see that for all M $\mathbb{N}$ we have

$$\lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q}||f_{n+v_M,1}\cdot f_{n+v_M,2}-f_{n,1}\cdot f_{n,2}||_{H_3}^2 \tag{2.140}$$

$$=\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||f_{n+v_M,1}\cdot f_{n+v_M,2}-f_{n,1}\cdot f_{n,2}||_{H_3}^2 \tag{2.141}$$

$$\leq 2\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||f_{n+v_M,1}\cdot f_{n+v_M,2}-f_{n+v_M,1}\cdot f_{n,2}||_{H_3}^2 \tag{2.142}$$

$$+2\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||f_{n+v_M,1}\cdot f_{n,2}-f_{n,1}\cdot f_{n,2}||_{H_3}^2$$

$$\leq 2\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}W^2C^2||f_{n+v_M,2}-f_{n,2}||_{H_2}^2 \tag{2.143}$$

$$+2\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}W^2C^2||f_{n+v_M,1}-f_{n,1}||_{H_1}^2\leq\frac{4W^2C^2}{2^M}.$$

□

Theorem 2.3.9. *Let H be a Hilbert space, $(x_n)_{n=1}^{\infty}$ $UB(H)$, and $(c_n)_{n=1}^{\infty}$ $UB(\mathbb{C})$.*

(i) If $(x_n)_{n=1}^{\infty}$ is a completely ergodic sequence and $(c_n)_{n=1}^{\infty}$ is an invariant sequence, then $(c_nx_n)_{n=1}^{\infty}$ is a completely ergodic sequence.

(ii) If $(x_n)_{n=1}^{\infty}$ H is a nearly weakly mixing sequence and $(c_n)_{n=1}^{\infty}$ is a compact sequence, then $(c_nx_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(iii) If $(x_n)_{n=1}^{\infty}$ is a nearly mildly mixing sequence and $(c_n)_{n=1}^{\infty}$ is a rigid sequence, then $(c_n x_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(iv) In each of (i), (ii), and (iii), we have that

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} c_n x_n|| = 0. \tag{2.144}$$

Proof of i. Let $(\xi_n)_{n=1}^{\infty}$ $UB(H)$ be any invariant sequence. By Lemma 2.3.8(i) we see that $(\overline{c_n}\xi_n)_{n=1}^{\infty}$ is an invariant sequence, so we see from Theorem 2.3.6(i) that

$$0 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \overline{c_n}\xi_n, x_n \ = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \xi_n, c_n x_n \ . \tag{2.145}$$

Since $(\xi_n)_{n=1}^{\infty}$ was arbitrary, we deduce from Theorem 2.3.6(i) that $(c_n x_n)_{n=1}^{\infty}$ is a completely ergodic sequence. □

Proof of ii. Let $(\xi_n)_{n=1}^{\infty}$ $UB(H)$ be any compact sequence. Since $(\overline{c_n})_{n=1}^{\infty}$ is a compact sequence, by Lemma 2.3.8(ii) we see that $(\overline{c_n}\xi_n)_{n=1}^{\infty}$ is a compact sequence. Since $(x_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence, we see from Theorem 2.3.6(ii) that

$$0 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \overline{c_n}\xi_n, x_n \ = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \xi_n, c_n x_n \ . \tag{2.146}$$

Since $(\xi_n)_{n=1}^{\infty}$ was an arbitrary compact sequence, we deduce from Theorem 2.3.6(ii) that $(c_n x_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence. □

Proof of iii. Let $(\xi_n)_{n=1}^{\infty}$ $UB(H)$ be any compact sequence. Since $(\overline{c_n})_{n=1}^{\infty}$ is a rigid sequence, by Lemma 2.3.8(iii) we see that $(\overline{c_n}\xi_n)_{n=1}^{\infty}$ is a rigid sequence. Since $(x_n)_{n=1}^{\infty}$ is a nearly mildly mixing sequence, we see from Theorem 2.3.6(iii) that

$$0 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n, \overline{c_n}\xi \ = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c_n x_n, \xi \ . \tag{2.147}$$

Since $(\xi_n)_{n=1}^{\infty}$ was an arbitrary compact sequence, we deduce from Theorem 2.3.6(ii) that $(c_n x_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence. □

Proof of iv. Since any of the sequences produced in parts (i), (ii), and (iii) are uniformly bounded and completely ergodic sequences, it suffices to apply Lemma 2.3.1(iii). □

Theorem 2.3.10. *Let* (X, B) *be a measurable space,* $\mu : B \to [0, \infty]$ *a measure,* $H = L^2(X, \mu)$*, and* $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \quad UB(H)$ *satisfying* $\sup_{n \quad \mathbb{N}} ||y_n||_{\infty} < \infty$*.*

(i) If $(x_n)_{n=1}^{\infty}$ *is a completely ergodic sequence and* $(y_n)_{n=1}^{\infty}$ *is an invariant sequence, then* $(y_n x_n)_{n=1}^{\infty}$ *is a completely ergodic sequence.*

(ii) If $(x_n)_{n=1}^{\infty}$ *is a nearly weakly mixing sequence and* $(y_n)_{n=1}^{\infty}$ *is a compact sequence, then* $(y_n x_n)_{n=1}^{\infty}$ *is a nearly weakly mixing sequence.*

(iii) If $(x_n)_{n=1}^{\infty}$ *is a nearly mildly mixing sequence and* $(y_n)_{n=1}^{\infty}$ *is a rigid sequence, then* $(y_n x_n)_{n=1}^{\infty}$ *is a nearly weakly mixing sequence.*

(iv) In each of (i), (ii), and (iii), we have that

$$\lim_{N \to \infty} ||\frac{1}{N} \sum_{n=1}^{N} y_n x_n|| = 0. \tag{2.148}$$

Proof. The proof of Theorem 2.3.10 is identical to that of Theorem 2.3.9 after realizing that $L^2(X, \mu)$ is a Hilbert space in which multiplication and complex conjugation are well defined, and that the inner product satisfies $\ x, yz\ =\ x\overline{y}, z\ $ for all $x, y, z \quad L^2(X, \mu)$. □

2.4 Applications to Uniform Distribution

2.4.1 Preliminaries

In this section we will often be working with sequences of complex numbers. Recalling that $\mathbb{C}$ is a Hilbert space when equipped with the inner product $\ z_1, z_2\ = z_1\overline{z_2}$, we will freely use the definitions of sections 1 and 2 in this context. For example, a sequence of complex numbers $(c_n)_{n=1}^{\infty}$ is a nearly strongly mixing sequence if it satisfies item (4) of definition 2.2.5 and $(c_n)_{n=1}^{\infty}$ is a compact sequence if it satisfies definition 2.3.3. Furthermore, throughout this section we will use m^d to denote the d-dimensional Lebesgue measure, and we may in some instances write $\exp(x)$ in place of $e^{2\pi i x}$ to increase readability if the expression replacing x is particularly long. Given vectors $\vec{k} = (k_1, \cdots, k_d) \quad \mathbb{Z}^d$ and $\vec{x} = (x_1, \cdots, x_d) \quad [0, 1]^d$ we may write $\ \vec{k}, \vec{x}\ := k_1 x_1 + \cdots + k_d x_d$. Before Discussing new results let us recall the definition of uniform distribution in $[0, 1]^d$ as well as some of its characterizations.

Definition 2.4.1. $(x_n)_{n=1}^{\infty} \quad [0, 1]^d$ *is a* ***uniformly distributed sequence*** *if*

$$\sup_{B \quad \mathcal{R}} \lim_{N \to \infty} \left| \frac{1}{N} |\{1 \le n \le N \mid x_n \quad B\}| - m^d(B) \right| = 0. \tag{2.149}$$

where $\mathcal{R}$ *denotes the collection of open rectangular prisms in* $[0, 1]^d$*.* $(x_n)_{n=1}^{\infty}$ *is* ***totally uniformly distributed*** *if for all* $a, b \quad \mathbb{N}$ *the sequence* $(x_{an+b})_{n=1}^{\infty}$ *is uniformly distrbuted.*

Theorem 2.4.2 (Weyl's Criterion)**.** *Given* $(x_n)_{n=1}^{\infty} \quad [0,1]^d$*, the following are equivalent.*

(i) $(x_n)_{n=1}^{\infty}$ *is uniformly distributed.*

(ii) for all $\vec{k} \quad \mathbb{Z}^d \setminus \{(0, \cdots, 0)\}$*, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \ \vec{k}, x_n} = 0. \tag{2.150}$$

(iii) for all $f \quad C([0,1]^d)$*, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{[0,1]^d} f dm^d. \tag{2.151}$$

We now record some definitions and facts regarding the notion of discrepancy. The discrepancy of a sequence is a measure of how far away the sequence is from being uniformly distributed. We will be intuitively thinking about the discrepancy of a sequence $(x_n)_{n=1}^{\infty}$ $[0,1]^d$ as the norm in a Hilbert space H whose vectors are sequences in $(y_n)_{n=1}^{\infty} \quad [0,1]^d$.

Definition 2.4.3. *Given a sequence* $(x_n)_{n=1}^{N} \quad [0,1]^d$*, the* ***discrepancy*** *of* $(x_n)_{n=1}^{\infty} \quad [0,1]^d$ *is denoted by* $D_N((x_n)_{n=1}^{N})$ *and given by*

$$D_N((x_n)_{n=1}^{N}) := \sup_{B \quad \mathcal{R}} \left| \frac{1}{N} |\{1 \le n \le N \mid x_n \quad B\}| - m^d(B) \right|, \tag{2.152}$$

where $\mathcal{R}$ *denotes the collection of all rectangular prisms contained in* $[0,1]^d$*. For an infinite sequence* $(x_n)_{n=1}^{\infty} \quad [0,1]^d$*, we let*

$$\overline{D}((x_n)_{n=1}^{\infty}) := \limsup_{N \to \infty} D_N((x_n)_{n=1}^{N}), \tag{2.153}$$

and we let

$$D((x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) := \lim_{q \to \infty} D_{N_q}((x_n)_{n=1}^{N_q}), \tag{2.154}$$

provided that the limit exists. The ***Isotropic discrepancy*** *of* $(x_n)_{n=1}^{N} \quad [0,1]^d$ *is*

$$J_N := \sup_{C \quad \mathcal{C}} \left| \frac{1}{N} |\{1 \le n \le N \mid x_n \quad C\}| - m^d(C) \right|, \tag{2.155}$$

where $\mathcal{C}$ *denotes the collection of all open convex subsets of* $[0,1]^d$*.*

It is worth noting that a sequence $(x_n)_{n=1}^{\infty} \in [0,1]^d$ is uniformly distributed if and only if $\overline{D}((x_n)_{n=1}^{\infty}) = 0$ (cf. Theorem 2.1.1 in [KN74]). Our next result compares D_N and J_N.

Theorem 2.4.4 (Theorem 2.1.6 in [KN74])**.** *For all* $(x_n)_{n=1}^{N} \in [0,1]^d$*, we have*

$$D_N((x_n)_{n=1}^{N}) \leq J_N((x_n)_{n=1}^{N}) \leq (4d^{\frac{3}{2}} + 1)D_N((x_n)_{n=1}^{N})^{\frac{1}{d}}. \tag{2.156}$$

Theorem 2.4.5 (Erdős-Turán-Koksma)**.** *For* $x_1, x_2, \cdots, x_N \in [0,1]^d$*, we have*

$$D_N(x_1, x_2, \cdots, x_N) \leq (\frac{3}{2})^d \left(\frac{2}{R+1} + \sum_{0<||r||_\infty \leq R} \frac{1}{m(r)} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \langle r, x_n \rangle} \right| \right), \tag{2.157}$$

where $m(r) = \prod_{i=1}^{d} \max(1, |r_i|)$ *for* $(r_1, r_2, \cdots, r_d) \in \mathbb{Z}^d$*.*

2.4.2 New Notions in the Theory of Uniform Distribution

Definition 2.4.6. *Let* $(x_n)_{n=1}^{\infty} \in [0,1]^d$ *and let* $\mathcal{C} = \{f \in C([0,1]^d \mid \int_{[0,1]^d} f dm^d = 0\}$*.*

(i) $(x_n)_{n=1}^{\infty}$ *is an* ***e-sequence*** *if for all* $f \in \mathcal{C}$, $(f(x_n))_{n=1}^{\infty}$ *is a completely ergodic sequence.*

(ii) $(x_n)_{n=1}^{\infty}$ *is an* ***wm-sequence*** *if for all* $f \in \mathcal{C}$, $(f(x_n))_{n=1}^{\infty}$ *is a nearly weakly mixing sequence.*

(iii) $(x_n)_{n=1}^{\infty}$ *is an* ***mm-sequence*** *if for all* $f \in \mathcal{C}$, $(f(x_n))_{n=1}^{\infty}$ *is a nearly mildly mixing sequence.*

(iv) $(x_n)_{n=1}^{\infty}$ *is an* ***sm-sequence*** *if for all* $f \in \mathcal{C}$, $(f(x_n))_{n=1}^{\infty}$ *is a nearly strongly mixing sequence.*

(v) $(x_n)_{n=1}^{\infty}$ *is an* ***o-sequence*** *if for all* $f \in \mathcal{C}$, $(f(x_n))_{n=1}^{\infty}$ *is a nearly orthogonal sequence.*

Lemma 2.3.1(iii) and Theorem 2.4.2(iii) show us that every e-sequence is a uniformly distributed sequence. Furthermore, we see that every wm-sequence is an e-sequence, every mm-sequence is a wm-sequence, every sm-sequence is a mm-sequence, and every o-sequence is a sm-sequence. An application of Lemma 2.3.1(i)-(ii) shows us that for items (i)-(iv) of Definition 2.4.6 it suffices to check that $(e^{2\pi i k x_n})_{n=1}^{\infty}$ is a sufficiently mixing sequence instead of checking for all $f \in \mathcal{C}$, but we will see in Theorem 2.4.28 that the same is not true for item (v). In Lemma 2.4 (3.1) of [Far21], it is shown that if $([0,1], \mathcal{B}, m, T)$ is a weakly (strongly) mixing measure preserving system (cf. Definitions 2.5.1 and 2.5.2), then for μ-a.e. $x \in [0,1]$, $(T^n x)_{n=1}^{\infty}$ is a wm-sequence (sm-sequence). We will also see in the coming subsections how e-sequences, wm-sequences, mm-sequences, and sm-sequences arise from variants of van der Corput's Difference Theorem as well. The case of o-sequences will also be examined (cf. Theorem 2.4.26). Before doing so, we will show that e-sequences, wm-sequences and mm-sequences are uniformly distributed along many subsequences in order to further illustrate how they are stronger properties than uniform distribution.

Definition 2.4.7. *For a sequence of natural numbers* $A = (n_k)_{k=1}^{\infty}$ *let*

$$\underline{d}(A) := \liminf_{N\to\infty} \frac{1}{N}|\{1 \le n \le N \mid n \quad A\}| = \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_A(n), \text{ and} \tag{2.158}$$

$$\overline{d}(A) := \limsup_{N\to\infty} \frac{1}{N}|\{1 \le n \le N \mid n \quad A\}| = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_A(n). \tag{2.159}$$

$\underline{d}(A)$ *is the* ***natural lower density of A*** *and* $\overline{d}(A)$ *is the* ***natural upper density of A****. If* $\overline{d}(A) = \underline{d}(A)$*, then we let* $d(A)$ *denote the common value which is the* ***natural density of A****.*

Definition 2.4.8. *Let* $B = (b_j)_{j=1}^{\infty}$ $\quad \mathbb{N}$ *be a strictly increasing sequence satisfying* $\underline{d}(B) > 0$*.*

(1) B *is a* ***invariant sequence*** *if*

$$d(B \quad (B-1)) = 0. \tag{2.160}$$

(2) B *is a* ***compact sequence*** *if for all* $\epsilon > 0$*, there exists* $k \quad \mathbb{N}$ *for which*

$$\sup_{m \quad \mathbb{N}} \min_{1\le k\le K} \overline{d}((B+m) \quad (B+k)) < \epsilon. \tag{2.161}$$

(3) B *is a* ***rigid sequence*** *if for all* $\epsilon > 0$*, there exists* $k \quad \mathbb{N}$ *for which*

$$\overline{d}((B+n_k) \quad B) < \epsilon. \tag{2.162}$$

We see that if $(b_j)_{j=1}^{\infty}$ *is an invariant, compact, or rigid sequence, then* $(\mathbb{1}_B(n))_{n=1}^{\infty}$ *is an invariant, compact, or rigid sequence (of complex numbers) respectively. The converse need not be true since we have simplified the definition of invariant, compact, and rigid sequences of natural numbers through the use of limit suprememums rather than checking a condition along a collection of subsequences. The condition that* $\underline{d}(B) > 0$ *intuitively tells us that the sequence will not be identified with* 0 *when we pass from H to H (or in this section, from* $\mathbb{C}$ *to a Hilbert spaces of sequences of complex numbers).*

Theorem 2.4.9. *Let* $(x_n)_{n=1}^{\infty} \subseteq [0,1]^d$ *and* $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ *be sequences.*

(i) If $(x_n)_{n=1}^{\infty}$ is an e-sequence if and only if for any invariant sequence $(n_k)_{k=1}^{\infty}$, $(x_{n_k})_{k=1}^{\infty}$ is a uniformly distributed sequence.

(ii) If $(x_n)_{n=1}^{\infty}$ is a wm-sequence if and only if for any compact sequence $(n_k)_{k=1}^{\infty}$, $(x_{n_k})_{k=1}^{\infty}$ is a uniformly distributed sequence.

(iii) If $(x_n)_{n=1}^{\infty}$ is a mm-sequence if and only if for any rigid sequence $(n_k)_{k=1}^{\infty}$, $(x_{n_k})_{k=1}^{\infty}$ is a uniformly distributed sequence.

Proof of the backwards directions. Let us first consider the case in which $(x_n)_{n=1}^{\infty}$ is an e-sequence. Let $m \in \mathbb{Z}^d \setminus \{\vec{0}\}$ be arbitrary, and note that $(e^{2\pi i m\cdot x_n})_{n=1}^{\infty}$ is a completely ergodic sequence. Letting $B = (n_k)_{k=1}^{\infty}$, we see that $(\mathbb{1}_B(n))_{n=1}^{\infty}$ is an invariant sequence, so by Theorem 2.3.6(i) we see that

$$0 = \lim_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^{N}e^{2\pi i m x_n}\mathbb{1}_B(n)\right| = \lim_{N\to\infty}\frac{1}{N}\left|\sum_{n_k\in[1,N]}e^{2\pi i m x_{n_k}}\right| \tag{2.163}$$
$$\geq \lim_{K\to\infty}\frac{\underline{d}(B)}{K}\left|\sum_{k=1}^{K}e^{2\pi i m x_{n_k}}\right|.$$

If $(x_n)_{n=1}^{\infty}$ is a wm-sequence (sm-sequence) then we repeat the above argument with the use of Theorem 2.3.6(ii) (Theorem 2.3.6(iii)) in place of Theorem 2.3.6(i). □

Proof of the forwards directions. The proof of this direction is much longer than the previous direction, so we will only give a complete proof for (ii). We will then outline the modifications needed to adapt the proof for (ii) to a proof for (i) and (iii). We begin with a useful lemma.

Lemma 2.4.10. *For a sequence $\Phi = (N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ let $\mathcal{A}_\Phi := \{A \subseteq \mathbb{N} \mid \mathbb{1}_A$ is a compact element of $\mathcal{H}(\mathbb{1}_A, \mathbb{1}_A, \Phi)\}$. If a sequence $(c_n)_{n=1}^{\infty} \in UB(\mathbb{C})$ is compact, then for any sequence $(M_q)_{q=1}^{\infty} \subseteq \mathbb{N}$, there is a subsequence $\Phi = (N_q)_{q=1}^{\infty}$ for which $(c_n)_{n=1}^{\infty}$ can be approximated arbitrarily closely by elements of $\text{Span}_{\mathbb{C}}(\{\mathbb{1}_A \mid A \in \mathcal{A}_\Phi\}) \subseteq \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space containing $\mathcal{H}((c_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ as a closed subspace.*

Proof of Lemma 2.4.10. By Lemma 3.3.1, let X be a compact metric space, $S : X \to X$ a continuous map, $F \in C(X)$ and $x \in X$ a point for which $F(S^n(x)) = c_n$. Let ν be any weak* limit point of $\{\frac{1}{M_q}\sum_{k=1}^{M_q}\delta_{S^k(x)}\}_{n=1}^{\infty}$ and let $\Phi = (N_q)_{q=1}^{\infty}$ be a subsequence for which $\{\frac{1}{N_q}\sum_{k=1}^{N_q}\delta_{S^k(x)}\}_{n=1}^{\infty}$ converges to ν in the weak* topology. Letting U denote the unitary operator induced by S and $\mathcal{H} = L^2(X,\mu)$, we see that $F \in \mathcal{H}_c$ since $(c_n)_{n=1}^{\infty}$ is a compact sequence. Let $\mathcal{K}$ denote the σ-algebra of the Kronecker factor $\mathcal{H}_c$, let $\epsilon > 0$ be arbitrary, and let $M = ||F||_\infty$. Let $L_1 = \{c \quad \mathsf{C} \mid \mathrm{Re}(\frac{c}{\epsilon}) \quad \mathsf{N}\}$ and $L_2 = \{c \quad \mathsf{C} \mid \mathrm{Im}(\frac{c}{\epsilon}) \quad \mathsf{N}\}$. Since ν is a probability measure, at most countably many members of $\{F^{-1}(L_1 + t)\}_{t \quad [0,\epsilon)}$ and of $\{F^{-1}(L_2 + ti)\}_{t \quad [0,\epsilon)}$ have positive measure, so $t_1, t_2 \quad [0, \epsilon)$ be such that $\nu(F^{-1}((L_1 + t_1) \quad (L_2 + t_2 i))) = 0$. Let $\{A_i\}_{i=1}^H$ be an enumeration of the connected components of $((L_1 + t_1) \quad (L_2 + t_2 i))^c$ that contain some $z \quad \mathsf{C}$ with $|z| \le M$. For $1 \le i \le H$, let $a_i \quad A_i$ be arbitrary, and note that for $g(y) = \sum_{i=1}^H a_i \mathbb{1}_{F^{-1}(A_i)}(y)$ we have $||F - g||_H \le ||F - g||_\infty < \epsilon$ and $F^{-1}(A_i) \quad \mathsf{K}$ for each $1 \le i < H$. Let $E_i = \{n \quad \mathsf{N} \mid S^n(x) \quad F^{-1}(A_i)\}$, and note that

$$\limsup_{N \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} |c_n - \sum_{i=1}^{H} a_i \mathbb{1}_{E_i}(n)|^2 \tag{2.164}$$
$$= \limsup_{N \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} |F(S^n(x)) - g(S^n(x))|^2 = ||F - g||_\infty^2 \le \epsilon^2.$$

It remains to check that $E_i \quad \mathsf{A}_\Phi$ for each $1 \le i \le H$. Since $\mathbb{1}_{F^{-1}(A_i)} \quad \mathsf{K}$ for each $1 \le i \le H$, let $K_i \quad \mathsf{N}$ be such that

$$\sup_{t \quad \mathsf{N}} \min_{1 \le k \le K_i} ||\mathbb{1}_{F^{-1}(A_i)} \circ S^k - \mathbb{1}_{F^{-1}(A_i)} \circ S^t||^2 < \epsilon. \tag{2.165}$$

Recalling that X is a compact metric space and $\{F^{-1}(A_i)\}_{i=1}^H$ is a collection of open sets, we see that

$$\sup_{t \quad \mathsf{N}} \min_{1 \le k \le K_i} \limsup_{N \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} |\mathbb{1}_{E_i}(n+k) - \mathbb{1}_{E_i}(n+t)|^2 \tag{2.166}$$
$$= \sup_{t \quad \mathsf{N}} \min_{1 \le k \le K_i} \limsup_{N \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} |\mathbb{1}_{F^{-1}(A_i)}(S^{n+k}(x)) - \mathbb{1}_{F^{-1}(A_i)}(S^{n+t}(x))|^2$$
$$= \sup_{t \quad \mathsf{N}} \min_{1 \le k \le K_i} ||\mathbb{1}_{F^{-1}(A_i)} \circ S^k - \mathbb{1}_{F^{-1}(A_i)} \circ S^t||_H^2 < \epsilon.$$

□

We now return to the proof of the forwards direction of (ii). We see from the comments after Definition 2.4.6 that $(x_n)_{n=1}^\infty$ is a wm-sequence if and only if for any $\vec{k} \quad \mathsf{Z}^d \setminus \{\vec{0}\}$ the sequence $(e^{2\pi i \vec{k} \cdot x_n})_{n=1}^\infty$ is a nearly weakly mixing sequence, so let us fix some $\vec{k} \quad \mathsf{Z}^d \setminus \{\vec{0}\}$. Due to Lemma 2.3.6(ii), it now suffices to show that $(e^{2\pi i \vec{k} \cdot x_n})_{n=1}^\infty$ is orthogonal to all compact sequences $(c_n)_{n=1}^\infty \quad \mathrm{UB}(\mathsf{C})$. Let $(M_q)_{q=1}^\infty \quad \mathsf{N}$ be any sequence for which $((e^{2\pi i \vec{k} \cdot x_n})_{n=1}^\infty, (c_n)_{n=1}^\infty, (M_q)_{q=1}^\infty$ is a weakly permissible triple and let $\epsilon > 0$ be arbitrary. Let $\Phi = (N_q)_{q=1}^\infty$, A_Φ, and H be as in Lemma 2.4.10. Let $g = \sum_{i=1}^H s_i \mathbb{1}_{E_i}$ be such that $||(c_n)_{n=1}^\infty - g||_H < \epsilon$ and $E_i \quad \mathsf{A}_\Phi$ for each $1 \le i \le H$. For each $1 \le i \le m$, let $E_{i,1} = E_i \quad 2\mathsf{N}$, and $E_{i,2} = E_i \cap (2\mathbb{N}+1)$, and note that $\underline{d}(E_{i,1}), \underline{d}(E_{i,2}) \geq \frac{1}{2}$. To see that $E_{i,1}$ and $E_{i,2}$ are compact sequences, let $\epsilon > 0$ be arbitrary, and let $K_i \quad \mathbb{N}$ be such that

$$\sup_{m \quad \mathbb{N}} \min_{1 \leq k \leq K_i} \overline{\lim_{N \to \infty}} \frac{1}{N} \sum_{n=1}^{N} |\mathbb{1}_{E_i}(n+k) - \mathbb{1}_{E_i}(n+m)| < \frac{\epsilon}{2}. \tag{2.167}$$

We now see that for $j \quad \{1, 2\}$ we have

$$\sup_{m \quad \mathbb{N}} \min_{\substack{1 \leq k \leq 2K_i \\ \text{k even}}} \overline{\lim_{N \to \infty}} \frac{1}{N} \sum_{n=1}^{N} |\mathbb{1}_{E_{i,j}}(n+k) - \mathbb{1}_{E_{i,j}}(n+m)| < \epsilon, \tag{2.168}$$

so $E_{i,1}$ and $E_{i,2}$ are indeed compact sequences. Letting $E_{i,j} = (n_{i,j,k})_{k=1}^{\infty}$ for (i,j) $[1,H] \times [1,2]$, and recalling that $(n)_{n=1}^{\infty} \quad \mathbb{N}$ is a compact sequence, we see that

$$\limsup_{N \to \infty} |\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \vec{k} \cdot w_n} c_n| \leq \epsilon + \limsup_{N \to \infty} |\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \vec{k} \cdot w_n} g(n)| \tag{2.169}$$

$$\leq \epsilon + \sum_{i=1}^{m} |s_i| \limsup_{N \to \infty} |\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \vec{k} \cdot w_n} \mathbb{1}_{E_i}(n)|$$

$$= \epsilon + \sum_{i=1}^{H} |s_i| \limsup_{N \to \infty} |\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i t \cdot w_n} (\mathbb{1}_{E_{i,1}}(n) + \mathbb{1}_{E_{i,2}}(n) - 1)|$$

$$\leq \epsilon + \sum_{i=1}^{H} |s_i| \limsup_{N \to \infty} \frac{1}{N} \left(|\sum_{n=1}^{N} e^{2\pi i \vec{k} \cdot w_n} \mathbb{1}_{E_{i,1}}(n)| + |\sum_{n=1}^{N} e^{2\pi i \vec{k} \cdot w_n} \mathbb{1}_{E_{i,2}}(n)| + |\sum_{n=1}^{N} e^{2\pi i \vec{k} \cdot w_n}| \right)$$

$$= \epsilon + \sum_{i=1}^{H} |s_i| \limsup_{N \to \infty} \frac{1}{N} \Big(|\sum_{n_{i,1,k} \quad [1,N]} e^{2\pi i t \cdot w_{n_{i,1,k}}}|$$

$$+ |\sum_{n_{i,2,k} \quad [1,N]} e^{2\pi i t \cdot w_{n_{i,2,k}}}| + |\sum_{n=1}^{N} e^{2\pi i t \cdot w_n}| \Big) = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the desired result follows.

To prove items (i) and (iii), we work with the σ-algebra of of invariant sets and the σ-algebra of sets that share the same rigidity sequence as $(c_n)_{n=1}^{\infty}$ (cf. Proposition 2.1 of [Ber+14]) respectively. We also replace the use of Lemma 2.3.6(ii) with that of Lemma 2.3.6(i) and 2.3.6(iii). Lastly, It is clear that if E_i are invariant sequences then $E_{i,1} := E_i \quad (\bigcup_{n \quad 2\mathbb{N}} [n^2, (n+1)^2 - 1)$ and $E_{i,2} := E_i \quad (\bigcup_{n \quad 2\mathbb{N}-1} [n^2, (n+1)^2 - 1)$ are also invariant sequences satisfying $\underline{d}(E_{i,1}), \underline{d}(E_{i,2}) \geq \frac{1}{2}$ and $\mathbb{1}_{E_i} = \mathbb{1}_{E_{i,1}} + \mathbb{1}_{E_{i,2}} - 1$. If E_i is a rigid sequence, then we recall that E_i is rigid along the IP-set generated by its rigidity sequence, hence we are still able to deduce that $E_{i,1}$ and $E_{i,2}$ are rigid sequences. □

2.4.3 An Ergodic van der Corput Difference Theorem

Before we can properly state the analog of Corollary 2.2.9 we need to recall some basic definitions and theorems regarding the uniform distribution of doubly indexed sequences.

Definition 2.4.11. $(x_{n,m})_{(n,m) \quad \mathbb{N}^2} \quad [0,1]^d$ *is* ***uniformly distributed*** *if for every open rectangular prism* $R \quad [0,1]^d$*, we have*

$$\lim_{K\to\infty} \sup_{N,M\geq K} \left| \frac{1}{NM} |\{(n,m) \quad [1,N]\times[1,M] \mid x_{n,m} \quad R\}| - \mu^d(R) \right| = 0. \tag{2.170}$$

Theorem 2.4.12. $(x_{(n,m)})_{(n,m) \quad \mathbb{N}^2} \quad [0,1]^d$ *is uniformly distributed if and only if for every* $\vec{k} \quad \mathbb{Z} \setminus \{(0,\cdots,0)\}$*, we have*

$$\lim_{K\to\infty} \sup_{N,M\geq K} \left| \frac{1}{NM} \sum_{(m,n) \quad [1,M]\times[1,N]} e^{2\pi i \; \vec{k}, x_{n,m}} \right| = 0. \tag{2.171}$$

Theorem 2.4.13. *If* $(x_n)_{n=1}^{\infty} \quad \mathbb{T}$ *is such that* $(x_{n+h} - x_n)_{(n,h) \quad \mathbb{N}^2}$ *is uniformly distributed, then* $(x_n)_{n=1}^{\infty}$ *is an e-sequence.*

We see that Theorem 2.4.13 is an immediate corollary to Corollary 2.2.9, Lemma 2.3.1(i)-(ii), and Theorem 2.4.12. We also note that for $\alpha \quad \mathbb{R} \setminus \mathbb{Q}$ the sequence $(n\alpha)_{n=1}^{\infty}$ satisfies Theorem 2.4.13 and is consequently an e-sequence. We will now construct in example 2.4.14 an e-sequence that is not totally uniformly distributed. Since any wm-sequence is totally uniformly distributed as a consequence of Theorem 2.4.9(ii), the sequence in example 2.4.14 is an example of an e-sequence that is not a wm-sequence. One can also use Theorem 2.4.18 of the next section to show that $(n\alpha)_{n=1}^{\infty}$ is not a wm-sequence.

Remark 2.4.14. Let $\alpha \quad \mathbb{R}\setminus\mathbb{Q}$ be arbitrary and let $L : [0,1) \to [0,\frac{1}{2})$ denote the map $x \to \frac{1}{2}x$. Let us consider the sequence $(y_n)_{n=1}^{\infty}$ given by $y_{2n} = L(2n\alpha)$ and $y_{2n-1} = L((2n-1)\alpha) + \frac{1}{2}$ for all $n \quad \mathbb{N}$. We see that $(y_n)_{n=1}^{\infty}$ is not totally uniformly distributed by construction. Now let us show that $(y_{n+h} - y_n)_{(n,h) \quad \mathbb{N}^2}$ is uniformly distributed. First, we note that

$$y_{n+h} - y_n = \begin{cases} L(h\alpha) & \text{if h is even and } n\alpha \quad [0,1-h\alpha) \\ L(h\alpha) + \frac{1}{2} & \text{if h is even and } n\alpha \quad [1-h\alpha,1) \\ L(h\alpha) + \frac{1}{2} & \text{if h is odd and } n\alpha \quad [0,1-h\alpha) \\ L(h\alpha) & \text{if h is odd and } n\alpha \quad [1-h\alpha,1) \end{cases}. \tag{2.172}$$

We remark that for each $h \quad \mathbb{N}$, the sequence $(y_{n+h} - y_n)_{n=1}^{\infty}$ takes on exactly 2 values. Now let $k \quad \mathbb{N}$ and $\epsilon > 0$ both be arbitrary and consider the functions

$$f_1(x) := (1 - x)e^{2\pi ikL(x)} + xe^{2\pi ik(L(x)+\frac{1}{2})} \text{ and} \tag{2.173}$$
$$f_2(x) := (1 - x)e^{2\pi ik(L(x)+\frac{1}{2})} + xe^{2\pi ikL(x)}.$$

We see that $f_1(x) + f_2(x) = 0$ if k is odd and $f_1(x) = 2e^{\pi ikx}$ if k is even, so $\int_0^1 (f_1(x) + f_2(x))dx = 0$ regardless of the value of k. Since $(2h\alpha)_{h=1}^{\infty}$ and $(2h\alpha - \alpha)_{h=1}^{\infty}$ are uniformly distributed, let $H_0 \quad \mathrm{N}$ be such that

$$\epsilon > \left|\int_0^1 f_1(x)dx - \frac{1}{2H}\sum_{h=1}^{H} f_1(2h\alpha)\right| \text{ and } \epsilon > \left|\int_0^1 f_2(x)dx - \frac{1}{2H}\sum_{h=1}^{H} f_2(2h\alpha - \alpha)\right| \tag{2.174}$$

whenever $H \geq H_0$. Since $(n\alpha)_{n=1}^{\infty}$ is uniformly distributed, we see that $\overline{D}((n\alpha)_{n=1}^{\infty}) = 0$, so let $N_0 \quad \mathrm{N}$ be such that $D_N((x_n)_{n=1}^{N}) < \epsilon$ for all $N \geq N_0$. We now see that if $N, H \geq \max(H_0, N_0)$, then

$$\frac{1}{2NH}\Big|\sum_{n=1}^{N}\sum_{h=1}^{2H} \mathrm{e}(2\pi ik(y_{n+h} - y_n))\Big| \tag{2.175}$$

$$= \frac{1}{2NH}\Big|\sum_{\substack{h \ [1,2H] \\ \text{h even}}}\left(\sum_{\substack{n \ [1,N] \\ n\alpha \ [0,1-h\alpha)}} \exp(k(y_{n+h} - y_n)) + \sum_{\substack{n \ [1,N] \\ n\alpha \ [1-h\alpha,1)}} \exp(k(y_{n+h} - y_n))\right) \tag{2.176}$$

$$+ \sum_{\substack{h \ [1,2H] \\ \text{h odd}}}\left(\sum_{\substack{n \ [1,N] \\ n\alpha \ [0,1-h\alpha)}} \exp(k(y_{n+h} - y_n)) + \sum_{\substack{n \ [1,N] \\ n\alpha \ [1-h\alpha,1)}} \exp(k(y_{n+h} - y_n))\right)\Big|$$

$$= \frac{1}{2NH}\Big|\sum_{\substack{h \ [1,2H] \\ \text{h even}}}\left(\sum_{\substack{n \ [1,N] \\ n\alpha \ [0,1-h\alpha)}} \exp(kL(h\alpha)) + \sum_{\substack{n \ [1,N] \\ n\alpha \ [1-h\alpha,1)}} \exp(k(L(h\alpha) + \frac{1}{2}))\right) \tag{2.177}$$

$$+ \sum_{\substack{h \ [1,2H] \\ \text{h odd}}}\left(\sum_{\substack{n \ [1,N] \\ n\alpha \ [0,1-h\alpha)}} \exp(k(L(h\alpha) + \frac{1}{2})) + \sum_{\substack{n \ [1,N] \\ n\alpha \ [1-h\alpha,1)}} \exp(kL(h\alpha))\right)\Big|$$

$$\leq 4\epsilon + \frac{1}{2H}\Big|\sum_{\substack{h \ [1,2H] \\ \text{h even}}}\Big((1 - \|h\alpha\|)\exp(kL(h\alpha)) + \|h\alpha\|\exp(k(L(h\alpha) + \frac{1}{2}))\Big) \tag{2.178}$$

$$+ \sum_{\substack{h \ [1,2H] \\ \text{h odd}}}\Big((1 - \|h\alpha\|)\exp(k(L(h\alpha) + \frac{1}{2})) + \|h\alpha\|\exp(kL(h\alpha))\Big)\Big|$$

$$=4\epsilon + \frac{1}{2H}\Big|\sum_{h=1}^{H}\Big((1-\|h\alpha\|)\exp(kL(2h\alpha)) + \|h\alpha\|\exp(k(L(2h\alpha)+\frac{1}{2}))\Big) \tag{2.179}$$

$$+\sum_{h=1}^{H}\Big((1-\|h\alpha\|)\exp(k(L(2h\alpha-\alpha)+\frac{1}{2})) + \|h\alpha\|\exp(kL(2h\alpha-\alpha))\Big)\Big|$$

$$\leq 5\epsilon + \frac{1}{2}\Big|\int_0^1 f_1(x)dx + \int_0^1 f_2(x)dx\Big| = 5\epsilon. \tag{2.180}$$

Example 2.1. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ be arbitrary, and consider the sequence $(y_n)_{n=1}^{\infty} \subseteq \mathbb{T}$ given by $y_n = m\alpha$ for $\binom{m}{2} < n \leq \binom{m+1}{2}$. $(y_n)_{n=1}^{\infty}$ is a sequence that is totally uniformly distributed in $[0,1]$ but is not an e-sequence. In fact, for every $f \in C([0,1])$ and $a,b \in \mathbb{N}$, $(f(y_{an+b}))_{n=1}^{\infty}$ is an invariant sequence. We leave it as an amusing exercise for the reader to show by direct computation that $(y_{n+h}-y_n)_{(n,h)\in\mathbb{N}^2}$ is not uniformly distributed.

2.4.4 Weak, Mild, and Strong Mixing van der Corput Difference Theorems

We will require the following technical lemma that relates strong Ceàro averages and natural density for use in some of the upcoming proofs.

Lemma 2.4.15. *For a bounded sequence of nonnegative real numbers $(x_n)_{n=1}^{\infty}$, we have*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}|x_h - L| = 0 \tag{2.181}$$

if and only if for every $\epsilon > 0$ we have

$$d(\{h \in \mathbb{N} \mid |x_h - L| < \epsilon\}) = 1. \tag{2.182}$$

We also require a definition analogous to that of a permissible triple from Section 2.2.

Definition 2.4.16. *Given a family $\{(x_{n,h})_{n=1}^{\infty}\}_{h=1}^{\infty}$ of sequences in $[0,1]^d$ and an increasing sequence $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$, we define $(\{(x_{n,h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ to be a **permissible pair** if for all $h \in \mathbb{N}$ $D((x_{n,h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is well defined.*

Theorem 2.4.17. *For $(x_n)_{n=1}^{\infty} \subseteq [0,1]^{d_1}$ the following are equivalent:*

(i) $(x_n)_{n=1}^{\infty}$ *is a wm-sequence.*

(ii) *For all uniformly distributed $(y_n)_{n=1}^{\infty} \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, y_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H} D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.183}$$

(iii) For all $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ *for which* $(\{(x_n, x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ *is a permissible pair, we have*

$$\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} D((x_n, x_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.184}$$

(iv) For all $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ *for which* $(\{(x_{n+h} - x_n)_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty}))$ *is a permissible pair, we have*

$$\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.185}$$

Proof. We will first show that (i) implies (ii). Let $(y_n)_{n=1}^{\infty} \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ be as in (ii). We will show that equation (2.183) holds by using Lemma 2.4.15. Let $\epsilon > 0$ be arbitrary, and let $R \in \mathbb{N}$ be such that

$$(\frac{3}{2})^d(\frac{2}{R+1}) < \frac{\epsilon}{2}. \tag{2.186}$$

Let $A = \{r \in \mathbb{Z}^{d_1+d_2} \mid 0 < ||r||_\infty \le R \ \&\ r_i = 0 \ \forall 1 \le i \le d_1.\}$ and $B = \{r \in \mathbb{Z}^{d_1+d_2} \mid 0 < ||r||_\infty \le R\} \setminus A$. Since $(y_n)_{n=1}^{\infty}$ is uniformly distributed, we note that for all $r \in A$ and $h \in \mathbb{N}$, we have

$$\lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q} e^{2\pi i\langle r,(x_n,y_{n+h})\rangle}\right| = \lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q} e^{2\pi i\langle (r_{d_1+1},\cdots,r_{d_1+d_2}),y_{n+h}\rangle}\right| = 0. \tag{2.187}$$

For each $r \in B$, we note that $(e^{2\pi i\langle (r_1,r_2,\cdots,r_d),x_n\rangle})_{n=1}^{\infty}$ is a nearly weakly mixing sequence. By Lemma 2.4.15, let

$$S_r = \Big\{h \in \mathbb{N} \mid \lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q}\exp(\langle (r_1,\cdots,r_{d_1}),x_n\rangle)\exp(\langle (r_{d_1+1},\cdots,r_{d_1+d_2}),y_{n+h}\rangle)\right| < \frac{\epsilon}{2(\frac{3}{2})^{d_1+d_2}(2R+1)^{d_1+d_2}}\Big\}, \tag{2.188}$$

where $(N_q)_{q=1}^{\infty}$ is any subsequence of $(N_q)_{q=1}^{\infty}$ for which all of the limits defining S_r exist. Since $d(S_r) = 1$ for every $r \in B$, we see that for $S := \cap_{r\in B} S_r$ we also have $d(S) = 1$. Furthermore, for every $h \in S$ and $r \in B$ we have

$$\lim_{q \to \infty} \left| \frac{1}{N_q} \sum_{n=1}^{N_q} \exp(\ r, (x_n, y_{n+h}) \ ) \right| < \frac{\epsilon}{2(\frac{3}{2})^{d_1+d_2}(2R+1)^{d_1+d_2}}. \quad (2.189)$$

We now see from Theorem 2.4.5 that $h \quad S$, we have

$$D((x_n, y_{n+h}), (N_q)_{q=1}^{\infty}) = \lim_{q \to \infty} D_{N_q}((x_n, y_{n+h})_{n=1}^{\infty}) \quad (2.190)$$

$$\leq \lim_{q \to \infty} (\frac{3}{2})^{d_1+d_2} \left(\frac{2}{R+1} + \sum_{0<||r||_{\infty} \leq R} \frac{1}{m(r)} \left| \frac{1}{N_q} \sum_{n=1}^{N_q} e^{2\pi i \ r, (x_n, y_{n+h})} \right| \right) \quad (2.191)$$

$$\leq \frac{\epsilon}{2} + \lim_{q \to \infty} \frac{1}{N_q} (\frac{3}{2})^{d_1+d_2} \left(\sum_{r \ A} + \sum_{r \ B} \right) \left| \frac{1}{N_q} \sum_{n=1}^{N_q} e^{2\pi i \ r, (x_n, y_{n+h})} \right| \leq \epsilon, \quad (2.192)$$

which concludes the proof that (i) implies (ii).

To show that (ii) implies (iii) it suffices to show that (ii) implies the uniform distribution of $(x_n)_{n=1}^{\infty}$. Let $(y_n)_{n=1}^{\infty} \quad [0,1]$ be an arbitrary uniformly distributed sequence and let $(N_q)_{q=1}^{\infty} \quad \mathbb{N}$ be arbitrary. By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality that $(\{(x_n, y_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair. We observe that if $R \quad [0,1]^{d_1}$ is an open rectanglular prism, then $R \times [0,1]$ is an open rectanglular prism in $[0,1]^{d_1+1}$, so $D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) \geq D((x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ for all $h \quad \mathbb{N}$. By Lemma 2.4.15 and the assumptions of (ii) we see that for each $\epsilon > 0$ there exists $h \quad \mathbb{N}$ for which $\epsilon > D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) \geq D((x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, which concludes the proof that (ii) implies (iii).

Now let us show that (iii) implies (iv). Let $T : [0,1]^{2d_1} \to [0,1]^{d_1}$ be the map defined by

$$T(x_1, \cdots, x_{d_1}, x_{d_1+1}, \cdots, x_{2d_1}) = (x_{d_1+1} - x_1, x_{d_1+2} - x_2, \cdots, x_{2d_1} - x_{d_1}) \pmod 1. \quad (2.193)$$

If $B \quad [0,1]^{d_1}$ is an open rectangle, then $T^{-1}B$ is a union of at most F open convex set in $[0,1]^{2d}$, where $F \quad \mathbb{N}$ is independent of B. It follows from Theorem 2.4.4 that for all open rectangle B, we have

$$D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) \leq F \cdot J((x_n, x_{n+h}), (N_q)_{q=1}^{\infty}) \quad (2.194)$$
$$\leq F(4(d_1 + d_2)^{\frac{3}{2}} + 1) D((x_n, x_{n+h}), (N_q)_{q=1}^{\infty})^{\frac{1}{d_1+d_2}},$$

so the desired result follows from Lemma 2.4.15.

Lastly, we prove that (iv) implies (i). Let $\epsilon > 0$ and $k \in \mathbb{N}$ both be arbitrary. Let $g(x) = \sum_{i=1}^{m} c_i \mathbb{1}_{B_i}(x)$ be a step function for which $||e^{2\pi ikx} - g(x)||_\infty < \frac{\epsilon}{2}$ and $||g(x)||_\infty \leq 1$. For each $h \in \mathbb{N}$, let $\gamma_h = D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$. We see that for $h \in H$, we have

$$\lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q} e^{2\pi ik(x_{n+h}-x_n)}\right| \leq \frac{\epsilon}{2} + \lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q} g(x_{n+h} - x_n)\right| \tag{2.195}$$

$$=\frac{\epsilon}{2} + \lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q}\sum_{i=1}^{m} c_i \mathbb{1}_{B_i}(x_{n+h} - x_n)\right|$$

$$\leq\frac{\epsilon}{2} + \left|\sum_{i=1}^{m} c_i\mu(B_i)\right| + \sum_{i=1}^{m}|c_i|\gamma_h = \frac{\epsilon}{2} + \sum_{i=1}^{m}|c_i|\gamma_h + \left|\int_0^1 g(x)dx\right| \leq \epsilon + \sum_{i=1}^{m}|c_i|\gamma_h.$$

From equation (2.185), we see that

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q} e^{2\pi ikx_{n+h}}\overline{e^{2\pi ikx_n}}\right| \leq \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\left(\epsilon + \sum_{i=1}^{m}|c_i|\gamma_h\right) = \epsilon. \tag{2.196}$$

Since $\epsilon > 0$ was arbitrary, Theorem 2.2.10 shows us that $(e^{2\pi ikx_n})_{n=1}^{\infty}$ is a nearly weakly mixing sequence. Since $k \in \mathbb{N}$ was also arbitrary, the desired result follows from Lemma 2.3.1(i)-(ii). □

Corollary 2.4.18. *Let $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ be a sequence for which*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{D}((x_{n+h} - x_n)_{n=1}^{\infty}) = 0. \tag{2.197}$$

Then $(x_n)_{n=1}^{\infty}$ is a wm-sequence.

Theorem 2.4.19. *For $(x_n)_{n=1}^{\infty} \subseteq [0,1]^{d_1}$ the following are equivalent:*

(i) $(x_n)_{n=1}^{\infty}$ *is a mm-sequence.*

(ii) *For all uniformly distributed $(y_n)_{n=1}^{\infty} \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, y_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have*

$$IP - \lim_{h\to\infty} D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.198}$$

(iii) For all $(N_q)_{q=1}^{\infty}$ N *for which* $(\{(x_n, x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ *is a permissible pair, we have*

$$IP - \lim_{h \to \infty} D((x_n, x_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.199}$$

(iv) For all $(N_q)_{q=1}^{\infty}$ N *for which* $((\{(x_{n+h} - x_n)_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty}))$ *is a permissible pair, we have*

$$IP - \lim_{h \to \infty} D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.200}$$

The proof of Theorem 2.4.19 is almost identical to that of Theorem 2.4.17 so we omit it.

Corollary 2.4.20. *Let* $(x_n)_{n=1}^{\infty}$ $[0,1]$ *be a sequence for which*

$$IP - \lim_{h \to \infty} \overline{D}((x_{n+h} - x_n)_{n=1}^{\infty}) = 0. \tag{2.201}$$

Then $(x_n)_{n=1}^{\infty}$ *is a mm-sequence.*

Theorem 2.4.21. *For* $(x_n)_{n=1}^{\infty}$ $[0,1]^{d_1}$ *the following are equivalent:*

(i) $(x_n)_{n=1}^{\infty}$ *is a sm-sequence.*

(ii) For all uniformly distributed $(y_n)_{n=1}^{\infty}$ $[0,1]^{d_2}$ *and* $(N_q)_{q=1}^{\infty}$ N *for which* $(\{(x_n, y_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ *is a permissible pair, we have*

$$\lim_{h \to \infty} D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.202}$$

(iii) For all $(N_q)_{q=1}^{\infty}$ N *for which* $(\{(x_n, x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ *is a permissible pair, we have*

$$\lim_{h \to \infty} D((x_n, x_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.203}$$

(iv) For all $(N_q)_{q=1}^{\infty}$ N *for which* $((\{(x_{n+h} - x_n)_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty}))$ *is a permissible pair, we have*

$$\lim_{h \to \infty} D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{2.204}$$

The proof of Theorem 2.4.21 is almost identical to that of Theorem 2.4.17 so we omit it.

Corollary 2.4.22. *Let* $(x_n)_{n=1}^{\infty} \quad [0,1]$ *be a sequence for which*

$$\lim_{h\to\infty} \overline{D}((x_{n+h} - x_n)_{n=1}^{\infty}) = 0. \tag{2.205}$$

Then $(x_n)_{n=1}^{\infty}$ *is a sm-sequence.*

Now let us compare the results of this subsection to another similar result from the literature.

Definition 2.4.23. *A sequence of natural numbers* $B = (n_k)_{k=1}^{\infty}$ *is a* ***Besicovitch Almost Periodic Sequence*** *if for all* $\epsilon > 0$*, there exists* $\alpha_1, \cdots, \alpha_k \quad [0,1]$ *and* $c_1, \cdots, c_k \quad$ *C for which*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\mathbb{1}_B(n) - \sum_{j=1}^{k} c_j e^{2\pi i \alpha_j n}| < \epsilon. \tag{2.206}$$

Theorem 2.4.24 (cf. Theorem 4.4 in [BM16])**.** *If* $(x_n)_{n=1}^{\infty} \quad [0,1]$ *is a sequence for which* $(x_{n+h} - x_n)_{n=1}^{\infty}$ *is uniformly distributed for every* $h \quad$ *N, then for all Besicovitch Almost Periodic Sequences* $(n_k)_{k=1}^{\infty}$*,* $(x_{n_k})_{k=1}^{\infty}$ *is uniformly distributed.*

Remark 2.4.25*.* Theorem 2.4.24 can also be deduced from section 4 of [DMF74]. Noting that any Besicovitch Almost Periodic Sequence of complex numbers is a compact sequence, and that any compact sequence of complex number is a rigid sequence, Theorem 2.4.9 shows us that Corollaries 2.4.18, 2.4.20, and 2.4.22 are each generalizations of Theorem 2.4.24. However, the astute reader may have noticed that we have yet to mention nearly orthogonal sequences in this section despite the apparent connection between the hypotheses of Theorems 2.1.2(i) and 2.4.24 with Corollary 2.2.17. The reason for this is that a sum of nearly orthogonal sequences is not necessarily a nearly orthogonal sequence, so we do not have an analogue of Lemma 2.3.1(i)-(ii) for nearly orthogonal sequences. Nonetheless, we may prove some Theorems in this direction as well.

2.4.5 Uniform Distribution and Orthogonality in Hilbert Spaces

Theorem 2.4.26. $(x_n)_{n=1}^{\infty} \quad [0,1]^d$ *is an* o*-sequence if and only if for each* $h \quad$ *N,* $(x_n, x_{n+h})_{n=1}^{\infty} \quad [0,1]^{2d}$ *is uniformly distributed.*

Proof. For the first direction, let us assume that $(x_n, x_{n+h})_{n=1}^{\infty}$ is uniformly distributed in $[0,1]^{2d}$ for all $h \quad \mathbb{N}$. We see that for all $k_1, k_2 \quad \mathbb{Z}^d$ that are not both $(0, 0, \cdots, 0)$ and any $h \quad \mathbb{N}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i(\; k_1, x_n \; + \; k_2, x_{n+h} \;)} = 0. \tag{2.207}$$

Now let $f \quad C([0,1]^d)$ satisfy $\int_{[0,1]^d} f dm^d = 0$ and let $(c_k)_{k \quad \mathbb{Z}^d}$ be the Fourier coefficients of f. Let $\epsilon \quad (0, ||f||_{\infty})$ be arbitrary, and let K be such that

$$||f(x) - \sum_{k \quad [-K,K]^d} c_k e^{2\pi i \; k, x} \; ||_{\infty} < \epsilon. \tag{2.208}$$

Noting that $c_{(0,0,\cdots,0)} = 0$, we see that for all $h \quad \mathbb{N}$ we have

$$\begin{aligned} &\lim_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(x_{n+h}) \overline{f(x_n)} \right| \qquad (2.209) \\ \leq &3\epsilon ||f||_{\infty} + \lim_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} (\sum_{k \quad [-K,K]^d} c_k e^{2\pi i \; k, x_{n+h}} \;)(\sum_{k \quad [-K,K]^d} c_k e^{-2\pi i \; k, x_n} \;) \right| \\ = &3\epsilon ||f||_{\infty} + \sum_{k_1, k_2 \quad [-K,K]^d} \lim_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} c_{k_1} \overline{c_{k_2}} e^{2\pi i(\; k_1, x_{n+h} \; + \; -k_2, x_n \;)} \right| = 3\epsilon ||f||_{\infty}. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we are done with the first direciton. For the reverse direction, let us assume that $(x_n)_{n=1}^{\infty}$ is an o-sequence. We will first show that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for all $h \quad \mathbb{N}$. To this end, let $k \quad \mathbb{Z}^d \setminus \{(0, 0 \cdots, 0)\}$ and $h \quad \mathbb{N}$ both be arbitrary and note that $(e^{2\pi i \; k, x_n} \;)_{n=1}^{\infty}$ is a nearly orthogonal sequence. Let $(N_q)_{q=1}^{\infty}$ be any sequence for which

$$\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} e^{2\pi i \; k, x_{n+h} - x_n} \tag{2.210}$$

exists. By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality that $((e^{2\pi i \; k, x_n} \;)_{n=1}^{\infty}, (e^{2\pi i \; k, x_n} \;)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple. Since $(e^{2\pi i \; k, x_n} \;)_{n=1}^{\infty}$ is a nearly orthogonal sequence it follows from Theorem 2.2.16 that

$$\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} e^{2\pi i \; k, x_{n+h} - x_n} \; = 0, \tag{2.211}$$

from which it follows that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is indeed uniformly distributed for all $h \quad \mathsf{N}$. Now let $h \quad \mathsf{N}$ be arbitrary, let $k_1, k_2 \quad \mathsf{Z}^d$ be such that k_1 and k_2 are not both $(0, 0, \cdots, 0)$ and let $(N_q)_{q=1}^{\infty} \quad \mathsf{N}$ be such that

$$\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} e^{2\pi i(\; k_1, x_n \; + \; k_2, x_{n+h} \;)} \tag{2.212}$$

exists. If k_1 or k_2 is $(0, 0, \cdots, 0)$, then the limit in equation 2.212 is 0 since the o-sequence $(x_n)_{n=1}^{\infty}$ is uniformly distributed, so let us assume that neither of k_1 and k_2 are $(0, 0, \cdots, 0)$. Note that for all $c \quad \mathsf{C}$ we have that $(e^{2\pi i \; k_1, x_n} + ce^{2\pi i \; k_2, x_n})_{n=1}^{\infty}$ is a nearly orthogonal sequence since $(x_n)_{n=1}^{\infty}$ is an o-sequence, so we once again see from Theorem 2.2.16 that

$$0 = \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} (e^{2\pi i \; k_1, x_{n+h}} + ce^{2\pi i \; k_2, x_{n+h}})(e^{-2\pi i \; k_1, x_n} + \overline{c}e^{-2\pi i \; k_2, x_n}) \tag{2.213}$$

$$= \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i \; k_1, x_{n+h} - x_n} + |c|^2 \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i \; k_2, x_{n+h} - x_n}$$

$$+ c \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_2, x_{n+h} \; - \; k_1, x_n \;)} + \overline{c} \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_1, x_{n+h} \; - \; k_2, x_n \;)}$$

$$= c \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_2, x_{n+h} \; - \; k_1, x_n \;)} + \overline{c} \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_1, x_{n+h} \; - \; k_2, x_n \;)}. \tag{2.214}$$

Letting $A(c)$ represent the quantity in (2.214), we observe that

$$0 = A(1) - iA(i) \tag{2.215}$$

$$= \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_2, x_{n+h} \; - \; k_1, x_n \;)} + \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_1, x_{n+h} \; - \; k_2, x_n \;)}$$

$$- i\left(i \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_2, x_{n+h} \; - \; k_1, x_n \;)} - i \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_1, x_{n+h} \; - \; k_2, x_n \;)} \right)$$

$$= 2 \lim_{q \to \infty} \frac{1}{N_q} \sum_{q=1}^{N_q} e^{2\pi i(\; k_2, x_{n+h} \; + \; -k_1, x_n \;)}.$$

Since $(k_1, k_2) \to (-k_1, k_2)$ is a bijection from $\mathsf{Z}^{2d} \setminus \left(\mathsf{Z}^d \times \{(0, \cdots, 0)\} \quad \{(0, \cdots, 0)\} \times \mathsf{Z}^d \right)$ to itself, we see that $(x_n, x_{n+h})_{n=1}^{\infty}$ is uniformly distributed for all $h \quad \mathsf{N}$. □

We would now like to that the sequences produced by Theorem 2.1.2 need not be o-sequences. To this end, we first require Lemma 2.4.27. While this Lemma 2.4.27 is well known, we use it repeatedly in the proof of Theorem 2.4.28, so we include a proof for the sake of completeness. For the rest of this section we will freely use the fact that if $p(x) = a_n x^n + \cdots + a_1 x + a_0 \quad \mathbb{R}[x]$ has a_i irrational for some $i \geq 1$, then $(p(n))_{n=1}^{\infty}$ (mod 1) is uniformly distributed (cf. Theorem 1.3.2 in [KN74]).

Lemma 2.4.27. *If $p(x) = a_n x^n + \cdots + a_1 x + a_0 \quad \mathbb{R}[x]$ is a non-constant polynomial for which a_i is irrational for some $i \geq 1$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i p(n)} \mathbb{1}_{2\mathbb{N}}(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i p(n)} \mathbb{1}_{2\mathbb{N}-1}(n) = 0. \tag{2.216}$$

Proof. We observe that $q_1(n) := p(2n), q_2(n) := p(2n-1) \quad \mathbb{R}[x]$ are non-constant polynomials with at least 1 irrational coefficient other than their constant coefficients. It now suffices to see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i p(n)} \mathbb{1}_{2\mathbb{N}}(n) = \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i q_1(n)} = 0, \text{ and} \tag{2.217}$$
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i p(n)} \mathbb{1}_{2\mathbb{N}-1}(n) = \frac{1}{2} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i q_2(n)} = 0$$

□

Theorem 2.4.28. *There exists a sequence $(x_n)_{n=1}^{\infty}$ such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every $h \quad \mathbb{N}$, but $(x_n)_{n=1}^{\infty}$ is not an o-sequence.*

Proof. Let $\alpha \quad \mathbb{R} \setminus \mathbb{Q}$ be arbitrary and consider the sequence $(x_n)_{n=1}^{\infty}$ defined by $x_n = n^2\alpha$ (mod 1) if n is odd and $x_n = 2(n-1)^2\alpha$ (mod 1) if n is even. To see that $(x_n)_{n=1}^{\infty}$ is not an o-sequence, it suffices by Theorem 2.4.26 to show that $(x_n, x_{n+1})_{n=1}^{\infty}$ is not uniformly distributed in $[0,1]^2$. To see that this is the case, we note that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(\ 2,-1 \ \cdot\ x_n, x_{n+1}\ ) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2x_n - x_{n+1}) \tag{2.218}$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(\mathbb{1}_{2\mathbb{N}-1}(n)\exp(2n^2\alpha - 2n^2\alpha) + \mathbb{1}_{2\mathbb{N}}(n)\exp(4(n-1)^2\alpha - (n+1)^2\alpha)\Big)$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(\mathbb{1}_{2\mathbb{N}-1}(n) + \mathbb{1}_{2\mathbb{N}}(n)\exp((3n^2 - 10n + 3)\alpha)\Big) = \frac{1}{2} = 0.$$

We will now show that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is a uniformly distributed sequence for every $h \quad \mathbb{N}$. If $h \quad \mathbb{N}$ is even then let $h = 2h$ and note that for all $k \quad \mathbb{N}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k(x_{n+h} - x_n)} \tag{2.219}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(\mathbb{1}_{2\mathsf{N}}(n) \exp(k(2(n + h \ - 1)^2 - 2(n - 1)^2)\alpha)$$

$$+ \mathbb{1}_{2\mathsf{N}-1}(n) \exp(k((n + h\ )^2 - n^2)\alpha) \Big)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(\mathbb{1}_{2\mathsf{N}}(n) \exp(k(4h\ n - 4n + h\ ^2)\alpha)$$

$$+ \mathbb{1}_{2\mathsf{N}-1}(n) \exp(k(2h\ n + h\ ^2)\alpha) \Big) = 0,$$

so $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every even h. If h N is odd then let $h = 2h\ + 1$ and note that for all k N we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k(x_{n+h} - x_n)} \tag{2.220}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(\mathbb{1}_{2\mathsf{N}}(n) \exp(k((n + h\ + 1)^2 - 2(n - 1)^2)\alpha)$$

$$+ \mathbb{1}_{2\mathsf{N}-1}(n) \exp(k(2(n + h\ )^2 - n^2)\alpha) \Big)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(\mathbb{1}_{2\mathsf{N}}(n) \exp(k(-n^2 + 2h\ n + 6n + h\ ^2 + 2h\ - 1)\alpha)$$

$$+ \mathbb{1}_{2\mathsf{N}-1}(n) \exp(k(n^2 + 4h\ n + 2h^2)\alpha) \Big) = 0,$$

so $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for all odd h as well. □

We also note that $(x_n, x_{n+h})_{n=1}^{\infty}$ is uniformly distributed in $[0,1]^2$ for all $h \geq 2$, but we omit the proof of this fact since we do not need it.

Definition 2.4.29. *For k N, a function $g : [0, \infty) \to$ R which is $(k+1)$-times continuously di erentiable is a* ***tempered function of order k*** *if the following hold.*

(1) $g^{(k+1)}(x)$ *tends monotonically to* 0 *as* x *tends to infinity.*

(2) $\lim_{x\to\infty} x g^{(k+1)}(x) = \infty$.

Theorem 2.4.30. *Let $g : [0, \infty) \to \mathbb{R}$ be a tempered function of order k that is $(k+2)$-times continuously di erentiable for some $k \geq 1$. Furthermore, suppose that there exists $B > 0$ for which $\limsup_{n\to\infty} |xg^{(k+2)}(x)| < B$.*

(i) If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, then $(g(n+h)\alpha, g(n+h), g(n)\alpha, g(n))_{n=1}^{\infty} \pmod 1$ is uniformly distributed in $\mathbb{T}^4$ for all $h \in \mathbb{N}$. In particular, $(g(n)\alpha, g(n))_{n=1}^{\infty} \pmod 1$ is an o-sequence in $[0,1]^2$ and $(g(n))_{n=1}^{\infty} \pmod 1$ is an o-sequence in $[0,1]$.

(ii) For all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$ we have that $(g(n+h) \alpha, g(n) \alpha)_{n=1}^{\infty} \pmod 1$ is uniformly distributed in its orbit closure.

Proof of (i). We begin by verifying that $(g(n+h)\alpha, g(n+h), g(n)\alpha, g(n))_{n=1}^{\infty} \pmod 1$ is uniformly distributed in $[0,1]^4$ for all $h \in \mathbb{N}$. To this end, let $(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 \setminus \{(0,0,0,0)\}$ be arbitrary so that we may show

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \exp(a_1 g(n+h)\alpha + a_2 g(n)\alpha + a_3 g(n+h) + a_4 g(n)) = 0. \tag{2.221}$$

It now suffices to show that $((a_1\alpha + a_3)g(n+h) + (a_2\alpha + a_4)g(n))_{n=1}^{\infty} \pmod 1$ is uniformly distributed in $[0,1]$, which will follow from showing that $h(x) := (a_1\alpha + a_3)g(x+h) + (a_2\alpha + a_4)g(x)$ is a tempered function. We begin by observing that

$$h^{(k+1)}(x) = (a_1\alpha + a_3)g^{(k+1)}(x+h) + (a_2\alpha + a_4)g^{(k+1)}(x) \underset{x\to\infty}{\longrightarrow} 0, \tag{2.222}$$

so condition (1) of Definition 2.4.29 holds. To see that condition (2) is also satisfied we consider 2 cases based on whether or not $(a_1, a_3) = (-a_2, -a_4)$. For our first case we assume that $(a_1, a_3) = (-a_2, -a_4)$ and by the Mean Value Theorem let $c = c(x) \in (x, x+h)$ be such that

$$g^{(k+1)}(x+h) - g^{(k+1)}(x) = g^{(k+2)}(c). \tag{2.223}$$

We now observe that

$$\begin{aligned} &|xh^{(k+1)(x)}| \qquad (2.224)\\ =&|(a_1\alpha + a_3)xg^{(k+1)}(x+h) + (a_2\alpha + a_4)xg^{(k+1)}(x)|\\ =&|(a_1\alpha + a_3)x(g^{(k+1)}(x+h) - g^{(k+1)}(x)) + (a_2\alpha + a_1\alpha + a_4 + a_3)xg^{(k+1)}(x)|\\ =&|(a_1\alpha + a_3)xg^{(k+2)}(c) + (a_2\alpha + a_1\alpha + a_4 + a_3)xg^{(k+1)}(x)|\\ \geq&|(a_2\alpha + a_1\alpha + a_4 + a_3)xg^{(k+1)}(x)| - |(a_1\alpha + a_3)B| \underset{x\to\infty}{\longrightarrow} \infty. \end{aligned}$$

For our second case we assume that $(a_1, a_3) = (-a_2, -a_4)$ and observe that $h(x) = (a_1\alpha + a_3)(g(x+h) - g(x))$ is a tempered function of order $(k-1)$, hence $(g(n+h)\alpha, g(n+h), g(n)\alpha, g(n))_{n=1}^{\infty}$ is uniformly distributed in T^4 for all $h \quad \mathrm{N}$.

The fact that $(g(n)\alpha, g(n))_{n=1}^{\infty} \pmod 1$ is an o-sequence now follows from Theorem 2.4.26. Similarly, after noting that the uniform distribution of $(g(n+h), g(n))_{n=1}^{\infty} \pmod 1$ is implied by that of $(g(n+h)\alpha, g(n+h), g(n)\alpha, g(n))_{n=1}^{\infty} \pmod 1$, we may again use Theorem 2.4.26 to see that $(g(n))_{n=1}^{\infty} \pmod 1$ is an o-sequence. □

Proof of (ii). Let us first consider the case in which $\alpha = \frac{r}{s}$ for coprime $r, s \quad \mathrm{Z} \setminus \{0\}$. Since r and s are coprime, it suffices to show the desired result for $\alpha = \frac{1}{s}$. Since $\frac{1}{s}g(x)$ also satisfies the hypoteses of Theorem 2.4.30, we see from part (i) that $(\frac{1}{s}g(n))_{n=1}^{\infty} \pmod 1$ is an o-sequence, hence $(\frac{1}{s}g(n+h), \frac{1}{s}g(n))_{n=1}^{\infty} \pmod 1$ is uniformly distributed in $[0,1]^2$ for all $h \quad \mathrm{N}$. It now suffices to observe that $(\frac{1}{s} \quad g(n+h) \quad , \frac{1}{s} \quad g(n)) \quad \pmod 1 = (\frac{i}{s}, \frac{j}{s})$ if and only if $(\frac{1}{s}g(n+h), \frac{1}{s}g(n)) \pmod 1 \quad [\frac{i}{s}, \frac{i+1}{s}) \times [\frac{j}{s}, \frac{j+1}{s})$.

We now consider the case in which $\alpha \quad \mathrm{R} \setminus \mathrm{Q}$ and proceed as in the proof of Lemma 5.12 from [BHK09]. We observe that for any $(a_1, a_2) \quad \mathrm{Z}^2 \setminus \{(0,0)\}$ the function $f(x_1, x_2, x_3, x_4) = \exp(a_1x_1 - a_1x_2\alpha + a_3x_3 - a_3x_4\alpha)$ is a Riemann Integrable function on $[0,1]^4$, hence we use the uniform distribution of $(g(n+h)\alpha, g(n+h), g(n)\alpha, g(n))_{n=1}^{\infty}$ (mod 1) to see that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \exp(a_1 \quad g(n+h) \quad \alpha + a_2 \quad g(n) \quad \alpha) \tag{2.225}$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\Big((g(n+h)\alpha, g(n+h), g(n)\alpha, g(n)) \pmod 1\Big) \tag{2.226}$$

$$= \int_{[0,1]^4} f d\vec{x} = 0. \tag{2.227}$$

□

Theorem 2.4.31. *If* $p(x) = a_nx^n + \cdots a_1x + a_0 \quad \mathrm{R}[x]$ *is a polynomial of degree* 2 *or more such that* a_i *is irrational for some* $i \geq 2$*, then* $(p(n))_{n=1}^{\infty} \pmod 1$ *is an o-sequence.*

Proof. We see that for any $(a, b) \quad \mathrm{Z}^2 \setminus \{(0,0)\}$ we have that $ap(n+h) + bp(n) \quad \mathrm{R}[x]$ has at least one irrational coefficient other than its constant coefficient, and is consequently a uniformly distributed sequence modulo 1. It follows that $(p(n), p(n+h))_{n=1}^{\infty} \pmod 1$ is uniformly distributed in $[0,1]^2$ for all $h \quad \mathrm{N}$, so the desired result follows from Theorem 2.4.26. □

2.5 Applications to Measure Preserving Systems

Definition 2.5.1. *Given a probability space $(X, \mathcal{B}, \mu)$ a measurable transformation $T : X \to X$ is called **measure preserving** if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$. A **measure preserving system (m.p.s.)** is a tuple $(X, \mathcal{B}, \mu, T)$ in which $(X, \mathcal{B}, \mu)$ is a probability space and $T : X \to X$ is a measure preserving transformation. For a m.p.s. $(X, \mathcal{B}, \mu, T)$ and $f \in L^2(X, \mu)$ we have $Tf(x) := f(Tx)$ and note that $T : L^2(X, \mu) \to L^2(X, \mu)$ is a unitary operator. A m.p.s. $(X, \mathcal{B}, \mu, T)$ is **rigid** if for every $f \in L^2(X, \mu)$ there exists $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ for which $T^{n_k} f \underset{k\to\infty}{\longrightarrow} f$ in the strong topology.*

We now wish to prove a theorem that demonstrates how the various notions of mixing sequences in Definition 2.2.5 can be used to characterize different levels of the ergodic hierarchy of mixing for a m.p.s., which we will now review.

Definition 2.5.2 (The Ergodic Hierarchy of Mixing). *Let $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ be a m.p.s.*

*(i) $\mathcal{X}$ is **ergodic** if for every $A \in \mathcal{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{-n}A \cap A) = \mu(A)^2. \tag{2.228}$$

*(ii) $\mathcal{X}$ is **totally ergodic** if $(X, \mathcal{B}, \mu, T^n)$ is ergodic for all $n \in \mathbb{N}$.*

*(iii) $\mathcal{X}$ is **weakly mixing** if for every $A \in \mathcal{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\mu(T^{-n}A \cap A) - \mu(A)^2| = 0. \tag{2.229}$$

*(iv) $\mathcal{X}$ is **mildly mixing** if for every $A \in \mathcal{B}$ we have*

$$IP-\lim_{n\to\infty} \mu(T^{-n}A \cap A) = \mu(A)^2. \tag{2.230}$$

*(v) $\mathcal{X}$ is **strongly mixing** if for every $A \in \mathcal{B}$ we have*

$$\lim_{n\to\infty} \mu(T^{-n}A \cap A) = \mu(A)^2. \tag{2.231}$$

Theorem 2.5.3. *Let $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ be a m.p.s.*

(i) X is ergodic if and only if for every $A \in \mathcal{B}$ we have that $(\mathbb{1}_{T^{-n}A} - \mu(A))_{n=1}^{\infty}$ is a completely ergodic sequence.

(ii) X is weakly mixing if and only if for every $A \in \mathcal{B}$ we have that $(\mathbb{1}_{T^{-n}A} - \mu(A))_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(iii) X is mildly mixing if and only if for every $A \in \mathcal{B}$ we have that $(\mathbb{1}_{T^{-n}A} - \mu(A))_{n=1}^{\infty}$ is a nearly mildly mixing sequence.

(iv) X is strongly mixing if and only if for every $A \in \mathcal{B}$ we have that $(\mathbb{1}_{T^{-n}A} - \mu(A))_{n=1}^{\infty}$ is a nearly strongly mixing sequence.

Proof. We first make the key observation that for all $h \in \mathbb{N}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle \mathbb{1}_{T^{-n-h}A} - \mu(A), \mathbb{1}_{T^{-n}A} - \mu(A)\rangle = \langle \mathbb{1}_{T^{-h}A} - \mu(A), \mathbb{1}_{A} - \mu(A)\rangle \quad (2.232)$$

$$= \mu(T^{-h}A \cap A) - \mu(A)^2.$$

To prove item (i) we observe that

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle \mathbb{1}_{T^{-n-h}A} - \mu(A), \mathbb{1}_{T^{-n}A} - \mu(A)\rangle \quad (2.233)$$

$$= \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\mu(T^{-h}A \cap A) - \mu(A)^2,$$

so Theorem 2.2.8 tells us that X is ergodic if and only if $(\mathbb{1}_{T^{-n}A})_{n=1}^{\infty}$ is completely ergodic for every $A \in \mathcal{B}$. The proofs of items (ii), (iii), and (iv) are similar and just replace the use of Theorem 2.2.8 with that of Theorems 2.2.10, 2.2.12, and 2.2.14 respectively. □

We will now provide a partial answer to a question that was asked by N. Frantzikinakis in [Fra22]. The question involves the notion of ***zero entropy*** which we have yet to discuss. We only require the reader to know that every rigid m.p.s. has zero entropy. We note that another answer is forthcoming in [FH].

Question 2.5.4 (cf. Problem 2 in [Fra22])**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving transformations. Suppose that the m.p.s. $(X, \mathcal{B}, \mu, T)$ has zero entropy and $f, g \in L^{\infty}(X, \mu)$.*

(i) Is it true that the averages

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{p(n)} g \tag{2.234}$$

converge in $L^2(X,\mu)$ when $p(n) = n$ or $p(n) = n^2$?

(ii) Is it true that for every $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ such that

$$\mu(A \cap T^{-n}A \cap S^{-p(n)}A) > 0 \tag{2.235}$$

when $p(n) = n$ or $p(n) = n^2$?

Theorem 2.5.5. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving transformations. Suppose that the m.p.s. $(X, \mathcal{B}, \mu, T)$ is rigid, and that the m.p.s. $(X, \mathcal{B}, \mu, S)$ is totally ergodic. Let $(k_n)_{n=1}^{\infty} \subset \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha \pmod 1)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$.*

(i) If $f, g \in L^\infty(X,\mu)$ are such that $\int_X g\,d\mu = 0$, then $(T^n f \cdot S^{k_n} g)_{n=1}^{\infty}$ is a nearly weakly mixing sequence in $L^2(X,\mu)$.

(ii) For any $f, g \in L^\infty(X,\mu)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{k_n} g = \mathbb{E}[f|\mathcal{I}_T] \int_X g\,d\mu, \tag{2.236}$$

where $\mathcal{I}_T = \{A \in \mathcal{B} \mid T^{-1}A = A\}$ is the σ-algebra of T-invariant sets and with convergence taking place in $L^2(X,\mu)$.

(iii) If $A_1, A_2, A_3 \in \mathcal{B}$ then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A_1 \cap T^{-n}A_2 \cap S^{-k_n}A_3) = \mu(A_3) \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A_1 \cap T^{-n}A_2). \tag{2.237}$$

(iv) If $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed in its orbit closure for all $\alpha \in \mathbb{R}$ then (i)-(iii) hold when $(X, \mathcal{B}, \mu, S)$ is ergodic.

For the proofs of (i)-(iv) all limits of sequences of vectors in $L^2(X,\mu)$ will be with respect to norm convergence in $L^2(X,\mu)$.

Proof of (i). We will use Corollary 2.2.17 to show that $(S^{k_n}g)_{n=1}^{\infty}$ is a nearly orthogonal sequence in H. Since $(\langle S^n g, g\rangle)_{n=1}^{\infty}$ is a positive definite sequence we may apply Bochner's Theorem and pick some positive finite measure ν on $[0,1]$ for which $\hat{\nu}(n) = \langle S^n g, g\rangle$ for all $n \in \mathbb{N}$. Since $(X, \mathcal{B}, \mu, S)$ is totally ergodic and $\int_X g d\mu = 0$ we see that $\nu(\mathbb{Q} \cap [0,1]) = 0$. We note that for all $h \in \mathbb{N}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle S^{k_{n+h}}g, S^{k_n}g\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle S^{k_{n+h}-k_n}g, g\rangle \tag{2.238}$$
$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \hat{\nu}(k_{n+h}-k_n) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int_0^1 e^{2\pi i(k_{n+h}-k_n)x} d\nu(x)$$
$$= \int_0^1 \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i(k_{n+h}-k_n)x} d\nu(x) = 0,$$

where the last equality follows from the fact that $((k_{n+h}-k_n)\alpha)_{n=1}^{\infty}$ (mod 1) is uniformly distributed for all $\alpha \in [0,1] \setminus \mathbb{Q}$ and $\nu(\mathbb{Q} \cap [0,1]) = 0$.

Since $(X, \mathcal{B}, \mu, T)$ is rigid, let $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ be such that $T^{n_k} f \xrightarrow[k\to\infty]{} f$ and observe that

$$\lim_{k\to\infty}\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} ||T^{n+n_k}f - T^n f|| = \lim_{k\to\infty} ||T^{n_k}f - f|| = 0, \tag{2.239}$$

so $(T^n f)_{n=1}^{\infty}$ is a rigid sequence in $UB(L^2(X,\mu))$. Since every nearly orthogonal sequence is a nearly mildly mixing sequence, we apply Theorem 2.3.10(iii) to see that $(T^n f \cdot S^{k_n} g)_{n=1}^{\infty}$ is a nearly weakly mixing sequence. □

Proof of (ii). Let $f, g \in L^\infty(X,\mu)$ be arbitrary. Let $\overline{g} = g - \int_X g d\mu$ so that $\int_X \overline{g}\, d\mu = 0$. We deduce from part (i) that $(T^n f \cdot S^{k_n} \overline{g})_{n=1}^{\infty} \subseteq UB(L^2(X,\mu))$ is a nearly weakly mixing sequence, hence we apply Lemma 2.3.1(iii) to see that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f \cdot S^{k_n} \overline{g} = 0. \tag{2.240}$$

We also see as a consequence of the Mean Ergodic Theorem that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f \int_X g d\mu = \mathbb{E}[f|\mathcal{I}_T] \int_X g d\mu. \tag{2.241}$$

It follows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\cdot S^{k_n}g \tag{2.242}$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\cdot S^{k_n}g\ +\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\int_X gd\mu = \mathsf{E}[f|I_T]\int_X gd\mu$$

□

Proof of (iii). By considering $f = \mathbb{1}_{A_2}$ we recall that we have shown in the proof of (i) that $(T^n\mathbb{1}_{A_2})_{n=1}^{\infty} = (\mathbb{1}_{T^{-n}A_2})_{n=1}^{\infty}$ is a rigid sequence. It is clear that $(\mathbb{1}_{A_1})_{n=1}^{\infty}$ is a compact sequence, so by Lemma 2.3.8(iii) we see that $(\mathbb{1}_{A_1}\cdot\mathbb{1}_{T^{-n}A_2})_{n=1}^{\infty} = (\mathbb{1}_{A_1\cap T^{-n}A_2})_{n=1}^{\infty}$ is a compact sequence. Applying part (i) with $f = 1 = \mathbb{1}_X$ and $g = \mathbb{1}_{A_3}$ we see that $(S^{k_n}\mathbb{1}_{A_3} - \mu(A_3))_{n=1}^{\infty}$ is a nearly weakly mixing sequence. It now follows from Lemma 2.3.6 that

$$0=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle\mathbb{1}_{S^{-k_n}A_3}-\mu(A_3),\mathbb{1}_{A_1\cap T^{-n}A_2}\rangle, \tag{2.243}$$

from which we deduce that

$$\mu(A_3)\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A_1\cap T^{-n}A_2)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle\mu(A_3),\mathbb{1}_{A_1\cap T^{-n}A_2}\rangle \tag{2.244}$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle\mathbb{1}_{S^{-k_n}A_3},\mathbb{1}_{A_1\cap T^{-n}A_2}\rangle=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A_1\cap T^{-n}A_2\cap S^{-k_n}A_3).$$

□

Proof of (iv). Since items (ii) and (iii) were proven as a result of item (i), it suffices to only show that item (i) in this new situation. To this end, it suffices to repeat the proof of (i) and observe that the measure ν given to us by Bochner's Theorem now satisfies $\nu(\{0,1\}) = 0$ instead of $\nu([0,1]\cap\mathbb{Q}) = 0$. Nonetheless, we see that the last equation of (2.238) will still hold since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(k_{n+h}-k_n)x}=0, \tag{2.245}$$

for all $x \in (0,1)$. □

Remark 2.5.6. Firstly, we observe that if $(k_n\alpha)_{n=1}^{\infty}$ (mod 1) is an o-sequence for every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, then $((k_{n+h}-k_n)\alpha)_{n=1}^{\infty}$ (mod 1) is uniformly distributed for all such α as a consequence of Theorem 2.4.26. Theorem 2.4.31 shows us that we can take $k_n = p(n)$ where

$p : \mathbb{Z} \to \mathbb{Z}$ is a polynomial of degree at least 2. It turns out that we can also take $k_n = n^\beta$ where $\beta \in (1, \infty) \setminus \mathbb{N}$ or $k_n = n^2 \log^2(n)$ as examples satisfying (iv) as is implied by Theorem 2.4.30. We note that Corollary 1.7 of [Fra22] does not apply to $k_n = n^2 \log^2(n)$.

Let us now recall a result of N. Frantzikinakis, E. Lesigne, and M. Wierdl (Theorem 1.4 and Corollary 4.4 of [FLW06]).

Theorem 2.5.7. *Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R = \{n \in \mathbb{N} \mid n^k\alpha \in [\frac{1}{4}, \frac{3}{4}]\}$.*

(i) If $(X, \mathcal{B}, \mu)$ is a probability space and $T_1, T_2, \cdots, T_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A \cap T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_{k-1}^{-n}A) > 0. \tag{2.246}$$

(ii) There exists a m.p.s. $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R$ we have

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) > 0. \tag{2.247}$$

Our next theorem strengthens the conclusion of Theorem 2.5.7(i). It is worth noting that the example of a m.p.s. $(X, \mathcal{B}, \mu, T)$ satisfying Theorem 2.5.7(ii) from [FLW06] has zero entropy.

Theorem 2.5.8. *Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R = \{n \in \mathbb{N} \mid n^k\alpha \in [\frac{1}{4}, \frac{3}{4}]\}$. Let $(X, \mathcal{B}, \mu)$ be a probability space and $S, T_1, T_2, \cdots, T_{k-1} : X \to X$ commuting invertible measure preserving transformations for which $(X, \mathcal{B}, \mu, S)$ is rigid. For any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which*

$$\mu(A \cap S^{-n}A \cap T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_{k-1}^{-n}A) > 0. \tag{2.248}$$

Proof. We remark that our proof is essentially the same as that in [FLW06] other than the fact that we use Corollary 2.2.17 in place of Theorem 2.1.2(i). We begin with a useful proposition.

Proposition 2.5.9. *Let $k \in \mathbb{N}$, $(X, \mathcal{B}, \mu)$ be a probability space, and $T_1, \cdots, T_{k-1} : X \to X$ be commuting measure preserving transformations. Let $p(x) \in \mathbb{R}_{\geq 0}[x]$ have degree at least*

k and an irrational leading coefficient, and let $g : \mathbb{T} \to \mathbb{C}$ be a Riemann integrable function satisfying $\int_{\mathbb{T}} g(x)dx = 0$. For any $f_1, \cdots, f_{k-1} \in L^\infty(X, \mu)$ the sequence

$$\left(T_1^n f_1 T_2^n f_2 \cdots T_{k-1}^n f_{k-1} g(p(n))\right)_{n=1}^\infty \tag{2.249}$$

is a nearly orthogonal sequence.

Proof of Proposition 2.5.9. By Lemma 2.3.1(i)-(ii) and standard approximation arguments we see that it suffices to prove the desired result for $g(x) = e^{2\pi i m x}$ with $m \in \mathbb{N}$. We now proceed by induction on k. For the base case of $k = 1$, we observe that $(n^k\alpha)_{n=1}^\infty$ is an o-sequence by Theorem 2.4.31, so $(e^{2\pi i m p(n)})_{n=1}^\infty \in \text{UB}(\mathbb{C})$ is a nearly orthogonal sequence, hence $(\mathbb{1}_X e^{2\pi i m p(n)})_{n=1}^\infty \in \text{UB}(L^2(X, \mu))$ is also a nearly orthogonal sequence. We now proceed to the inductive step and will show that the desired result holds for $k+1$ assuming that it holds for k. Let $a_n = T_1^n f_1 T_2^n f_2 \cdots T_k^n f_k e^{2\pi i m p(n)}$, $g_{i,h} = T_i^h f_i \overline{f_i}$, $\tilde{T}_i = T_i T_1^{-1}$, and observe that for each $h \in \mathbb{N}$ we have

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle a_{n+h}, a_n \rangle \qquad (2.250)\\ =&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X T_1^n g_{1,h} T_2^n g_{2,h} \cdots T_k^n g_{k,h} e^{2\pi i m (p(n+h) - p(n))} d\mu \\ =&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X g_{1,h} \tilde{T}_2^n g_{2,h} \cdots \tilde{T}_k^n g_{k,h} e^{2\pi i m (p(n+h) - p(n))} d\mu = 0, \end{aligned}$$

where the last equality follows from the inductive hypothesis and Lemma 2.3.6. The fact that $(a_n)_{n=1}^\infty$ is a nearly orthogonal sequence now follows from Corollary 2.2.17. □

Returning to the proof of Theorem 2.5.8, we observe that $g(x) = \mathbb{1}_{[\frac{1}{4},\frac{3}{4}]}(x) - \frac{1}{2}$ is a Riemann integrable function satisfying $\int_{\mathbb{T}} g(x)dx = 0$. We also observe that $(\mathbb{1}_{A\cap S^{-n}A})_{n=1}^\infty \in \text{UB}(L^2(X, \mu)$ is rigid. Since the nearly orthogonal sequence in equation (2.249) with $f_i = \mathbb{1}_A$ for all i is also a nearly mildly mixing sequence, it follows from Lemma 2.3.6(iii) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap S^{-n}A \cap T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_{k-1}^{n}A) e^{2\pi i m n^k \alpha} = 0, \tag{2.251}$$

for any $m \in \mathbb{N}$. We now see that

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[\frac{1}{4},\frac{3}{4}]}(n^k\alpha) \mu(A \cap S^{-n}A \cap T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_{k-1}^{n}A) \qquad (2.252)\\ =&\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{2} \mu(A \cap S^{-n}A \cap T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_{k-1}^{n}A) > 0, \end{aligned}$$

where the last inequality follows from the multiple recurrence of Furstenberg and Katznelson [FK78]. □

2.6 Comparison of Notions of Mixing Sequences

In this section we will compare our definitions of completely ergodic, nearly weakly mixing, and nearly strong mixing sequences with those of ergodic, weakly mixing, and strongly mixing sequences appearing in [BB86].

Definition 2.6.1 (cf. Definition 1.1 in [BB86]). *A bounded sequence $(f_n)_{n=1}^{\infty}$ H is* ***strongly mixing*** *if and only if for all $\epsilon > 0$ there exists K N such that for all m N the inequality $| \; f_n, f_m \; | \geq \epsilon$ has at most K solutions n.*

Theorem 2.6.2. *Let $(f_n)_{n=1}^{\infty}$ H be a bounded sequence. If $(f_n)_{n=1}^{\infty}$ is a strongly mixing sequence then it is also a nearly strongly mixing sequence.*

Proof. Let us assume for the sake of contradiction that $(f_n)_{n=1}^{\infty}$ is a strongly mixing sequence but not a nearly strongly mixing sequence. Without loss of generality, we may assume that $(f_n)_{n=1}^{\infty}$ satisfies $||f_n|| \leq 1$ for all n. Let $(N_q)_{q=1}^{\infty}$ N be such that $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ be a permissible triple. Since $(f_n)_{n=1}^{\infty}$ is not a nearly strongly mixing sequence, by Theorem 2.2.14 there exists $\epsilon > 0$ and $(h_k)_{k=1}^{\infty}$ N for which

$$\epsilon \leq \lim_{q\to\infty}\left|\frac{1}{N_q}\sum_{n=1}^{N_q} \; f_{n+h_k}, f_n \; \right| \leq \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} | \; f_{n+h_k}, f_n \; | \tag{2.253}$$

for every k N. Since $(f_n)_{n=1}^{\infty}$ is a strongly mixing sequence, let K N be such that for all m N the inequality $| \; f_n, f_m \; | \geq \frac{\epsilon}{2}$ has at most K solutions in n. For each k N, let

$$B_k = \{n \quad \mathrm{N} \mid | \; f_{n+h_k}, f_n \; | \geq \frac{\epsilon}{2}\}. \tag{2.254}$$

By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality that

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} \mathbb{1}_{B_k}(n) \tag{2.255}$$

exists for every k N. We see that

$$\begin{aligned}
\epsilon &\leq \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} | \ f_{n+h_k}, f_n \ | &(2.256)\\
&\leq \lim_{q\to\infty} \frac{1}{N_q}\left(\sum_{n \quad B_k\cap[1,N_q]} | \ f_{n+h_k}, f_n \ | + \sum_{n \quad B_k^c\cap[1,N_q]} | \ f_{n+h_k}, f_n \ |\right)\\
&\leq \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q}\left(\mathbb{1}_{B_k}(n) + \frac{\epsilon}{2}(1-\mathbb{1}_{B_k}(n)\right)\\
&=\frac{\epsilon}{2} + \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q}(1-\frac{\epsilon}{2})\mathbb{1}_{B_k}(n), \text{ hence}\\
&\lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q}\mathbb{1}_{B_k}(n) \geq \frac{\epsilon}{2-\epsilon} \geq \frac{\epsilon}{2}.
\end{aligned}$$

Let M N be such that $\frac{3M\epsilon}{10} \geq K$, and let Q be such that

$$\frac{1}{N_Q}\sum_{n=1}^{N_Q} | \ f_{n+h_k}, f_n \ | \geq \frac{9\epsilon}{10} \tag{2.257}$$

for every $1 \leq k \leq M$. We see from equation (2.257) that

$$\sum_{(n,k) \quad [1,N_Q]\times[1,M]} | \ f_{n+h_k}, f_n \ | \geq \frac{9MN_Q\epsilon}{10}. \tag{2.258}$$

We see from the definition of K that

$$\begin{aligned}
\sum_{(n,k) \quad [1,N_Q]\times[1,M]} | \ f_{n+h_k}, f_n \ &= \sum_{n=1}^{N_Q}\sum_{k=1}^{M} | \ f_{n+h_k}, f_n \ | \leq \sum_{n=1}^{N_Q}(K+(M-K)\frac{\epsilon}{2}) &(2.259)\\
&=KN_Q + (M-K)N_Q\frac{\epsilon}{2} \leq \frac{3MN_Q\epsilon}{10} + \frac{MN_Q\epsilon}{2} = \frac{8MN_Q\epsilon}{10},
\end{aligned}$$

which contradicts equation (2.258) and yields the desired result. □

Luckily, it is much easier to construct an example of a sequence $(f_n)_{n=1}^{\infty}$ H that is nearly strongly mixing but not strongly mixing. To see that this is the case, let H be any infinite dimensional Hilbert space and let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis of H. We see that $(e_n)_{n=1}^{\infty}$ is certainly a strongly mixing sequence so it is also a nearly strongly mixing sequence. Let $(f_n)_{n=1}^{\infty}$ be defined by $f_n = e_1$ if $n = 2^m$ for some m N and $f_n = e_n$ for all other n. We see that $(f_n)_{n=1}^{\infty}$ is not a strongly mixing sequence but still satisfies equation (2.52) and is therefore a nearly strongly mixing sequence by Theorem 2.2.14.

Definition 2.6.3 (cf. Definition 3.2 in [BB86]). *A bounded sequence* $(f_n)_{n=1}^{\infty}$ H *is* ***weakly mixing*** *if and only if for all* $\delta, \epsilon > 0$ *there exists* L $\mathbb{N}$ *such that for every* $N \geq L$ *and* m $\mathbb{N}$ *the inequality* $| \; f_n, f_m \; | \geq \epsilon$ *has at most* δN *solutions* n *with* $1 \leq n \leq N$.

Theorem 2.6.4. *Let* $(f_n)_{n=1}^{\infty}$ H *be a bounded sequence. If* $(f_n)_{n=1}^{\infty}$ *is a weakly mixing sequence then it is also a nearly weakly mixing sequence.*

Proof. Let us assume for the sake of contradiction that $(f_n)_{n=1}^{\infty}$ is a weakly mixing sequence but not a nearly weakly mixing sequence. Without loss of generality, we may assume that $(f_n)_{n=1}^{\infty}$ satisfies $||f_n|| \leq 1$ for all n. Let $(N_q)_{q=1}^{\infty}$ $\mathbb{N}$ be such that $((f_n)_{n=1}^{\infty}, (f_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ be a permissible triple. Since $(f_n)_{n=1}^{\infty}$ is not a nearly weakly mixing sequence, by Theorem 2.2.10 and Lemma 2.4.15 there exists $\epsilon > 0$ and $(h_k)_{k=1}^{\infty}$ $\mathbb{N}$ with $\overline{d}((h_k)_{k=1}^{\infty}) > 0$ for which

$$\epsilon \leq \lim_{q \to \infty} \left| \frac{1}{N_q} \sum_{n=1}^{N_q} \; f_{n+h_k}, f_n \; \right| \leq \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} | \; f_{n+h_k}, f_n \; | \tag{2.260}$$

for every k $\mathbb{N}$. Since $(f_n)_{n=1}^{\infty}$ is a weakly mixing sequence, let L $\mathbb{N}$ be such that for all m $\mathbb{N}$ the inequality $| \; f_n, f_m \; | \geq \frac{\epsilon}{2}$ has at most $\frac{\epsilon}{3}N$ solutions in n. For each k $\mathbb{N}$, let

$$B_k = \{n \quad \mathbb{N} \mid | \; f_{n+h_k}, f_n \; | \geq \frac{\epsilon}{2}\}. \tag{2.261}$$

By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary, we may assume without loss of generality that

$$\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \mathbb{1}_{B_k}(n) \tag{2.262}$$

exists for every k $\mathbb{N}$. As we saw in the proof of Theorem 2.6.2, we have

$$\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \mathbb{1}_{B_k}(n) \geq \frac{\epsilon}{2}. \tag{2.263}$$

Let M $\mathbb{N}$ be such that $\frac{3M\epsilon}{10} \geq K$, and let Q be such that $N_Q \geq L$, $h_M \leq \frac{\epsilon}{20}N_Q$, and

$$\frac{1}{N_Q} \sum_{n=1}^{N_Q} | \; f_{n+h_k}, f_n \; | \geq \frac{9\epsilon}{10} \tag{2.264}$$

for every $1 \leq k \leq M$. We see from equation (2.264) that

$$\sum_{(n,k) \quad [1,N_Q]\times[1,M]} | \; f_{n+h_k}, f_n \; | \geq \frac{9MN_Q\epsilon}{10}. \tag{2.265}$$

We see from the definition of L that

$$\sum_{(n,k) \quad [1,N_Q]\times[1,M]} | \; f_{n+h_k}, f_n \; = \sum_{n=1}^{N_Q}\sum_{k=1}^{M} | \; f_{n+h_k}, f_n \; | \tag{2.266}$$

$$= \sum_{n=1}^{N_Q-h_M}\sum_{k=1}^{M} | \; f_{n+h_k}, f_n \; | + \sum_{n=N_Q-h_M+1}^{N_q}\sum_{k=1}^{M} | \; f_{n+h_k}, f_n \; |$$

$$\leq \sum_{n=1}^{N_Q-h_M}\left(M\frac{\epsilon}{3} + (M - M\frac{\epsilon}{3})\frac{\epsilon}{2}\right) + h_M M$$

$$= M(N_Q - h_M)(\frac{\epsilon}{3} + (1 - \frac{\epsilon}{3})\frac{\epsilon}{2}) + h_M M \leq MN_Q\frac{5\epsilon}{6} + MN_Q\frac{\epsilon}{20} < \frac{9MN_Q\epsilon}{10},$$

which contradicts equation (2.265) and yields the desired result. □

We will now construct an example of a nearly strongly mixing sequence that is not even a weakly mixing sequence. We first need to recall one more Theorem from [BB86].

Theorem 2.6.5 (cf. Theorem 3.1 in [BB86]). *A bounded sequence $(f_n)_{n=1}^{\infty}$ is weakly mixing if and only if for all sequence $A = (n_k)_{k=1}^{\infty}$ with $\underline{d}(A) > 0$ we have*

$$\lim_{k\to\infty} ||\frac{1}{k}\sum_{j=1}^{k} f_{n_j}|| = 0. \tag{2.267}$$

For an example of a nearly strongly mixing sequence $(f_n)_{n=1}^{\infty}$ that is not even a weakly mixing sequence, note that for all $\alpha \quad \mathbb{R} \setminus \mathbb{Q}$, $(e^{2\pi i n^2\alpha})_{n=1}^{\infty}$ is a nearly strongly mixing sequence by Theorem 2.2.14. Furthermore, $(n^2\alpha \pmod 1)_{n=1}^{\infty}$ is a sm-sequence by Theorem 2.4.22, so it is also uniformly distributed. It follows that for

$$A = \{n \quad \mathbb{N} \mid |e^{2\pi i n^2\alpha} - 1| < \frac{1}{2}\} \tag{2.268}$$

we have $d(A) > 0$. Letting $A = (n_k)_{k=1}^{\infty}$, we see that

$$\text{Re}(\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k} e^{2\pi i n_k^2\alpha}) = \lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k}\text{Re}(e^{2\pi i n_k^2\alpha}) \geq \frac{1}{2}, \tag{2.269}$$

so $(e^{2\pi i n^2\alpha})_{n=1}^{\infty}$ is not a weakly mixing sequence by Theorem 2.6.5. It is interesting to note that nearly strongly mixing sequences exist in finite dimensional Hilbert spaces, but weakly mixing sequences do not.

Definition 2.6.6 (cf. Definition 3.1 in [BB86]). *A bounded sequence $(f_n)_{n=1}^{\infty}$ H is **ergodic** if and only if*

$$\lim_{N\to\infty}\frac{1}{N}||\sum_{n=1}^{N}f_n|| = 0. \tag{2.270}$$

As a result of item (iii) of lemma 2.3.1 we see any any completely ergodic sequence is also an ergodic sequence. For an example of an ergodic sequence that is not completely ergodic consider $(f_n)_{n=1}^{\infty}$ $\mathbb{C}$ given by

$$1,-1,1,1,-1,-1,1,1,1,-1,-1,-1,\cdots\underbrace{1,\cdots,1,}_{n}\underbrace{-1,\cdots,-1,}_{n}\cdots \tag{2.271}$$

It is clear that $(f_n)_{n=1}^{\infty}$ is an ergodic sequence that is also an invariant sequence. To see that $(f_n)_{n=1}^{\infty}$ is not a completely ergodic sequence we use the invariance of $(f_n)_{n=1}^{\infty}$ to see that

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g_{n+h}, g_n \;=\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g_n, g_n \;= 1. \tag{2.272}$$

It follows from Theorem 2.2.8 that $(f_n)_{n=1}^{\infty}$ is not a completely ergodic sequence.

2.7 IP Sets and Ultrafilters

Definition 2.7.1. *Given a set S the collection of finite subsets of S is denoted by $\mathcal{P}_f(S)$. Given $(n_k)_{k=1}^{\infty}$ $\mathbb{N}$ the set of finite sums of $(n_k)_{k=1}^{\infty}$ is $FS(n_k)_{k=1}^{\infty} := \{\sum_{k\ \alpha} n_k\}_{\alpha\ \mathcal{P}_f(\mathbb{N})}$. A $\mathbb{N}$ is an **IP-set** if $FS(n_k)_{k=1}^{\infty}$ A for some $(n_k)_{k=1}^{\infty}$ $\mathbb{N}$. B $\mathbb{N}$ is an **IP -set** if $A\cap B =$ whenever A is an IP-set.*

Definition 2.7.2. *Given a Hausdor topological space X and a sequence $(x_n)_{n=1}^{\infty}$ X we have that*

$$IP\ -\lim_{n\to\infty}x_n = x, \tag{2.273}$$

if x X is such that $\{n$ $\mathbb{N}\ |\ x_n$ $U\}$ is an IP -set for every open neighborhood U of x.

Definition 2.7.3. *Let* $P(\mathbb{N})$ *denote the power set of* $\mathbb{N}$*.* $p \quad P$ *is an* ***ultrafilter*** *if it satisfies the following conditions:*

(i) $\quad / p$.

(ii) If $A \quad p$ *and* $B \quad A$ *then* $B \quad p$.

(iii) If $A, B \quad p$ *then* $A \cap B \quad p$.

(iv) For every $A \quad \mathbb{N}$ *we have that either* $A \quad p$ *or* $A^c \quad p$.

If p *only satisfies properties (i)-(iii) then* p *is a* ***filter***.

See [HS12] for a comprehensive introduction to ultrafilters and some of their applications. We will only review a some facts about taking limits along filters for use in the proof of Theorem 2.3.6(iii). To this end, let X be a Hausdorff topological space, $p \quad P(\mathbb{N})$ a filter, and let $x, x_1, x_2, \cdots, x_n, \cdots \quad X$ be such that for every open neighborhood of U of x we have $\{n \quad \mathbb{N} \mid x_n \quad U\} \quad p$. In this situation we write $p - \lim_{n\to\infty} x_n = x$. If X is a compact Hausdorff topological space p is an ultrafilter then for any $x_1, x_2, \cdots, x_n, \cdots \quad X$ there exists a unique $x \quad X$ for which $p - \lim_{n\to\infty} x_n = x$.

IP-sets are related to a special type of ultrafilter known as an idempotent ultrafilter. Since we do not wish to discuss the fine details of the algebra of ultrafilters, we omit the classical definition of idempotent ultrafilter in favor of the following equivalent definition: An ultrafilter $p \quad P(\mathbb{N})$ is an **idempotent ultrafilter** if for every compact Hausdorff topological space X and every sequence $x_1, x_2, \cdots, x_n, \cdots \quad X$ we have

$$p - \lim_{m\to\infty} p - \lim_{n\to\infty} x_{n+m} = p - \lim_{n\to\infty} x_n. \tag{2.274}$$

Another important fact that we will use is that for any sequence $(n_k)_{k=1}^{\infty} \quad \mathbb{N}$ there exists an idempotent ultrafilter p for which $\text{FS}(n_k)_{k=1}^{\infty} \quad p$ (cf. Theorem 5.12 in [HS12]).

We conclude this section with a proof of a well known lemma that is used in the proof of Lemma 2.3.8(iii).

Definition 2.7.4. $A \quad \mathbb{N}$ *is a* ***Bohr***$_0$***-set*** *if there exists* $K \quad \mathbb{N}$*,* $\lambda_1, \cdots, \lambda_K \quad \mathbb{S}^1$*, and* $\epsilon > 0$ *such that* $\{h \quad \mathbb{N} \mid |1 - \lambda_i^h| < \epsilon \quad 1 \le i \le K\} \quad A$.

Lemma 2.7.5. *If* $A \quad \mathbb{N}$ *is a Bohr*$_0$*-set then* A *is IP .*

Proof. Let $k \quad \mathsf{N}$, $\lambda_1, \cdots, \lambda_K \quad \mathsf{S}^1$, $(n_k)_{k=1}^{\infty} \quad \mathsf{N}$, and $\epsilon > 0$ all be arbitrary. It suffices to find $n \quad \mathrm{FS}(n_k)_{k=1}^{\infty}$ for which $|1 - \lambda_i^n| < \epsilon$ for all $1 \leq i \leq K$. Let $s_j = \sum_{k=1}^{j} n_k$ for all $j \quad \mathsf{N}$. Let $R = \{(c_1, \cdots, c_K) \quad \mathsf{C}^K \mid |c_i| \leq 1 \quad 1 \leq i \leq K\}$ and observe that R is totally bounded, so there is some $N \quad \mathsf{N}$ for which $(\lambda_1^{s_j}, \cdots, \lambda_K^{s_j})_{j=1}^{N} \quad R$ contains 2 distinct points $(\lambda_1^{s_{j_1}}, \cdots, \lambda_K^{s_{j_1}})$ and $(\lambda_1^{s_{j_2}}, \cdots, \lambda_K^{s_{j_2}})$ whose distance is at most ϵ. Since

$$\epsilon \geq \left|(\lambda_1^{s_{j_2}}, \cdots, \lambda_K^{s_{j_2}}) - (\lambda_1^{s_{j_1}}, \cdots, \lambda_K^{s_{j_1}})\right| = \left|(\lambda_1^{s_{j_2} - s_{j_1}}, \cdots, \lambda_K^{s_{j_2} - s_{j_1}}) - (1, \cdots, 1)\right|, \quad (2.275)$$

it remains only to observe that $s_{j_2} - s_{j_1} = \sum_{k=j_1+1}^{j_2} n_k \quad \mathrm{FS}(n_k)_{k=1}^{\infty}$. □

CHAPTER 3

POINTWISE ERGODIC THEOREMS FOR HIGHER LEVELS OF MIXING

This chapter uses filters and filter convergence (cf. Definition 2.7.3) to obtain generalizations of the results in [Far21] for Hilbert space-valued functions and other levels of the ergodic hierarchy of mixing.

3.1 Introduction

In this section we establish the notation that we use, review some known results related to Birkhoff's Ergodic Theorem and state the main theorems of this chapter. The main results will be proven for Hilbert space-valued functions (cf. Section 3.2), but we will discuss them in this section for complex-valued functions for the sake of concreteness. We remark that some vector-valued ergodic theorems have previously been obtained in [Cha62] and [HST78], but our results are of a different nature (cf. Corollary 3.3.6). Whenever we discuss a measure preserving system (m.p.s.) $(X, \mathcal{B}, \mu, T)$, X will be a measurable space, $\mathcal{B}$ will be a σ-algebra of subsets of X, μ will be a probability measure on $(X, \mathcal{B})$ and $T : X \to X$ will be a measurable transformation satisfying $\mu(A) = \mu(T^{-1}A)$ for all $A \in \mathcal{B}$. When we work with the Hilbert space $L^2(X, \mu)$, we will let $U : L^2(X, \mu) \to L^2(X, \mu)$ denote the unitary operator given by $U(f) = f \circ T$. When we work with a m.p.s. of the form $([0, 1], \mathcal{B}, \mu, T)$, we will assume that $\mathcal{B}$ is the completion of the Borel σ-algebra. We say that two sequences of complex numbers $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are orthogonal if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} x_n \overline{y_n} = 0. \tag{3.1}$$

The classical ergodic hierarchy of mixing properties for a m.p.s. is introduced in Definition 2.5.2, so let us instead introduce a different version of the ergodic hierarchy of mixing through the use of filters..

Definition 3.1.1. *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $p \in \mathcal{P}(\mathbb{N})$ be a filter. The m.p.s. $(X, \mathcal{B}, \mu, T)$ is **p-mixing** if for all $A, B \in \mathcal{B}$ we have $p - lim_{n \to \infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$.*

Remark 3.1.2. We see that if $p = p_D$ is the filter of sets with natural density 1 (cf. Definition 2.4.7) then we recover the notion of weak mixing. If $p = p_{\mathrm{IP}}$ is the filter of IP -sets (cf. Definition 2.7.1) then we recover the notion of mild mixing. If $p = p_c$ is the filter of cofinite sets then we recover the notion of strong mixing. We may also obtain notions of mixing other than those of Definition 2.5.2 by considering filters such as idempotent ultrafilters. Interestingly, the notions of ergodicity, K-mixing, and Bernoullicity cannot be recovered from an appropriate choice of filter p. Let us now examine some of the existing pointwise ergodic theorems to develop some context for our generalizations.

Theorem 3.1.3 (Birkhoff, [Bir31])**.** *Let* (X, B, μ, T) *be a m.p.s., and let* $f \quad L^1([0,1], \mu)$*. For a.e.* $x \quad X$*, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x) = f\ (x), \tag{3.2}$$

where $f\ (x) \quad L^1(X,\mu)$ *is such that* $f\ (Tx) = f\ (x)$ *for a.e.* $x \quad X$ *and* $\int_A f\ d\mu = \int_A f d\mu$ *for every* $A \quad B$ *satisfying* $A = T^{-1}(A)$*. In particular, if* T *is ergodic, then for a.e.* $x \quad X$ *we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x) = \int_X f d\mu. \tag{3.3}$$

Remark 3.1.4. Birkhoff's Ergodic Theorem can be interpreted as follows. Given an ergodic m.p.s. (X, B, μ, T) and $f \quad L^1(X,\mu)$ satisfying $\int_X f d\mu = 0$, the sequence $(f(T^n x))_{n=1}^{\infty}$ is orthogonal to the constant sequence $(1)_{n=1}^{\infty}$. Since the transformation T can be viewed as acting on the sequence $(f(T^n x))_{n=1}^{\infty}$ by a left shift, and the sequence $(1)_{n=1}^{\infty}$ is invariant under the left shift, Birkhoff's Ergodic Theorem is an instance of the duality between ergodicity and invariance. The Wiener-Wintner Theorem is a generalization of Birkhoff's Ergodic Theorem for weakly mixing systems and has a similar interpretation.

Theorem 3.1.5 (Wiener-Wintner,[WW41])**.** *Let* (X, B, μ, T) *be a m.p.s. and let* f $L^1(X,\mu)$*. There exists* $X \quad B$ *with* $\mu(X\ ) = 1$*, such that for every* $x \quad X$ *and any* $\lambda \quad \mathbb{C}$ *with* $|\lambda| = 1$ *the limit*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x)\lambda^n \tag{3.4}$$

exists. Furthermore, if T *is weakly mixing, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x)\lambda^n = \begin{cases} 0 & \textit{if } \lambda = 1 \\ \int_X f d\mu & \textit{if } \lambda = 1 \end{cases}. \tag{3.5}$$

Remark 3.1.6. The Wiener-Wintner Theorem can be interpreted as follows. Given a weakly mixing m.p.s. $(X, \mathcal{B}, \mu, T)$ and $f \quad L^1(X, \mu)$ satisfying $\int_X f d\mu = 0$, the sequence $(f(T^n x))_{n=1}^{\infty}$ is orthogonal to any Besicovitch Almost Periodic Sequence $(y_n)_{n=1}^{\infty}$ (cf. Definition 2.4.23). Recalling that we can view T as acting on $(f(T^n x))_{n=1}^{\infty}$ by a left shift and that any Besicovitch Almost Periodic Sequence $(y_n)_{n=1}^{\infty}$ has a pre-compact orbit (under a topology that has not yet been specified) under the left shift, the Wiener-Wintner Theorem is an instance of the duality between weak mixing and compactness. The purpose of this chapter is to generalize Birkhoff's Ergodic Theorem, the Wiener-Wintner Theorem, and to prove new pointwise ergodic theorems through the use of dualities similar to the previously mentioned ones. To do so, we first require some definitions analogous to those in Definition 2.2.5.

Definition 3.1.7. *Let H be a Hilbert space.*

(i) The collections of averageable sequences $A(H)$, uniformly averageable sequences $UA(H)$, and uniformly bounded sequences $UB(H)$ be given by

$$A(H) := \Big\{ (x_n)_{n=1}^{\infty} \quad H \mid \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1} \|x_n\| < \infty \Big\}, \tag{3.6}$$

$$UA(H) := \Big\{ (x_n)_{n=1}^{\infty} \quad A(H) \mid \quad \epsilon > 0 \quad M > 0 \text{ s.t.}$$
$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \|x_n\| \mathbb{1}_{\|x_n\| > M}(n) < \epsilon \Big\}, \text{ and}$$
$$UB(H) := \Big\{ (f_n)_{n=1}^{\infty} \quad H \mid \sup_{n \quad \mathbb{N}} \|f_n\| < \infty \Big\}.$$

(ii) For an increasing sequence $(N_q)_{q=1}^{\infty} \quad \mathbb{N}$, $(x_n)_{n=1}^{\infty} \quad A(H)$, and $(y_n)_{n=1}^{\infty} \quad UB(H)$ we say that $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a ***weakly permissible triple*** *if*

$$\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \quad x_{n+h}, y_n \tag{3.7}$$

exists for all $h \quad \mathbb{N}$.

(iii) $(x_n)_{n=1}^{\infty} \quad A(H)$ is ***fully ergodic*** *if for all weakly permissible triples $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have*

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \quad x_{n+h}, y_n \quad = 0. \tag{3.8}$$

*(iv) If $p \quad P(\mathbb{N})$ is a filter, then $(x_n)_{n=1}^{\infty} \quad A(H)$ is **almost p-mixing** if for all weakly permissible triples $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have*

$$p - \lim_{h\to\infty} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \quad x_{n+h}, y_n \quad = 0. \tag{3.9}$$

*(v) $(y_n)_{n=1}^{\infty} \quad UB(H) \quad A(H)$ is **invariant** if for all weakly permissible triples $((y_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have*

$$\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||y_{n+1} - y_n|| = 0. \tag{3.10}$$

*Equivalently, $(y_n)_{n=1}^{\infty} \quad UB(H)$ is **invariant** if*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} ||y_{n+1} - y_n|| = 0. \tag{3.11}$$

*(vi) If $p \quad P(\mathbb{N})$ is a filter, then $(y_n)_{n=1}^{\infty} \quad UB(H) \quad A(H)$ is **almost p-rigid** if for all weakly permissible triples $((y_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ we have*

$$p - \lim_{h\to\infty} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||y_{n+h} - y_n|| = 0. \tag{3.12}$$

We refer the reader to Lemma 2.2.20 to see why we have to work with the inconvenient notion of weakly permissible triples. We remark that the sequences appearing in Definition 3.1.7(i)-(ii) are intuitively the same as the sequences appearing in Definition 2.2.5. The main difference is that in Chapter 2 we were working with Hilbert spaces such as $L^2(X, \mu)$ which required square averageable sequences, but in this chapter we seek pointwise ergodic theorems for functions in $L^1(X, \mu)$ which require averageable sequences. While $A(H)$ is the natural generalization of $SA(H)$, we will see that sequences produced by our pointwise ergodic theorems actually reside in $UA(H)$, and this observation will be crucial for our applications. We also observe that the definition of invariant sequences here coincides with that in Definition 2.3.2 for elements of $UB(H)$. Now let us examine some of the dualities that arise from Definition 3.1.7.

Lemma 3.1.8 (cf. Lemma 3.2.4)**.** *Suppose that $(x_n)_{n=1}^{\infty} \quad UA(\mathbb{C})$ and $(y_n)_{n=1}^{\infty} \quad UB(\mathbb{C})$.*

(i) If $(x_n)_{n=1}^{\infty}$ *is fully ergodic and* $(y_n)_{n=1}^{\infty}$ *is invariant then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n\overline{y_n}=0. \tag{3.13}$$

(ii) Suppose that p_1, p_2 $\mathcal{P}(\mathbb{N})$ *are filters such that for every* A_1 p_1 *and* A_2 p_2 *we have* $A_1 \cap A_2 =$ *. If* $(x_n)_{n=1}^{\infty}$ *is* p_1*-mixing and* $(y_n)_{n=1}^{\infty}$ *is* p_2*-rigid then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n\overline{y_n}=0. \tag{3.14}$$

Theorems 3.1.9 and 3.1.10 are corollaries of two of the main results of this chapter and are generalizations of the Birkhoff Pointwise Ergodic Theorem and the Wiener-Wintner Theorem respectively.

Theorem 3.1.9 (cf. Theorem 3.3.5 in Section 3)**.** *Let* $(X, \mathcal{B}, \mu, T)$ *be an ergodic m.p.s. and let* f $L^1(X,\mu)$ *satisfy* $\int_X f d\mu = 0$*. For a.e.* x X*,* $(f(T^n x))_{n=1}^{\infty}$ *is a fully ergodic sequence.*

Theorem 3.1.10 (cf. Theorem 3.3.8 in Section 3)**.** *Let* p $\mathcal{P}(\mathbb{N})$ *be a filter, let* $(X, \mathcal{B}, \mu, T)$ *be a* p*-mixing m.p.s., and let* f $L^1(X,\mu)$*. For a.e.* x X*,* $(f(T^n x))_{n=1}^{\infty}$ *is almost* p*-mixing.*

Since $(1)_{n=1}^{\infty}$ is an invariant sequence, Remark 3.1.4 and Lemma 3.1.8 show us that Theorem 3.1.9 is a generalization of Birkhoff's Ergodic Theorem. Similarly, we let p_1 denote the filter of sets of natural density 1, p_2 the filter of Bohr$_0$ sets, and observe that every Besicovitch Almost Periodic Sequence is p_2-rigid, so Remark 3.1.6 and Lemma 3.1.8 show us that Theorem 3.1.10 is a generalization of the Wiener-Wintner Theorem. To see that the class of p_2-rigid sequences is strictly larger than the class of Besicovitch Almost Periodic Sequences it suffices to note that every invariant sequence (such as $x_n = y_m$ for all $\binom{m}{2} < n \le \binom{m+1}{2}$, where $(y_m)_{m=1}^{\infty}$ is arbitrary) is p_2-rigid, but not every Besicovitich Almost Periodic Sequence is invariant (consider $x_n = (-1)^n$). The next main result of this paper is an analogue of Theorem 3.1.9 for strongly mixing measure preserving systems.

To give context to the last main result of this chapter let us consider Proposition 3.1.11, which gives a partial converse to Birkhoff's Ergodic Theorem. Proposition 3.1.11 is well known and an easy consequence of the Dominated Convergence Theorem.

Proposition 3.1.11. *Let* $(X, \mathcal{B}, \mu, T)$ *be a m.p.s. If for every* f $L^{\infty}(X,\mu)$*, there exists* A_f $\mathcal{B}$ *such that* $\mu(A_f) = 1$ *and for every* x A_f *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(T^n x) = \int_X f d\mu, \tag{3.15}$$

then T is ergodic.

Proposition 3.1.12 is a converse to the Wiener-Wintner Theorem in the same fashion that Proposition 3.1.11 is a converse to Birkhoff's Ergodic Theorem.

Proposition 3.1.12. *If for every f $L^\infty(X,\mu)$ with $\int_X f d\mu = 0$ there exist a set A_f X satisfying $\mu(A_f) = 1$ and for every x A_f equation (3.5) is satisfied, then T is weakly mixing.*

Proof. Let us recall that an ergodic m.p.s. $(X, \mathcal{B}, \mu, T)$ is weakly mixing if and only if $L^2(X,\mu)$ has no non-constant eigenfunctions with respect to U. Since any m.p.s. $(X, \mathcal{B}, \mu, T)$ satisfying the hypothesis of Proposition 3.1.12 also satisfies the hypothesis of Proposition 3.1.11 we may assume without loss of generality that $(X, \mathcal{B}, \mu, T)$ is ergodic. Now let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. that is ergodic but not weakly mixing and let f $L^2(X,\mu)$ be an eigenfunction corresponding to an eigenvalue $\lambda = 1$. We see that for a.e. x X we have

$$0 = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(T^n x)\lambda^{-n} = f(x), \tag{3.16}$$

hence $f = 0$. □

Theorem 3.1.13 is a converse to Theorem 3.1.10 in the same way that Propositions 3.1.11 and 3.1.12 are converses for Birkhoff's Ergodic Theorem and the Wiener-Wintner Theorem respectively.

Theorem 3.1.13 (cf. Theorem 3.3.10 in Section 3)**.** *Let p $\mathcal{P}(\mathbb{N})$ be a filter and $(X, \mathcal{B}, \mu, T)$ a m.p.s.Pfor every f $L^\infty(X,\mu)$ with $\int_X f d\mu = 0$ there exist a set A_f X satisfying $\mu(A_f) = 1$ and for every x A_f we have that $(f(T^n x))_{n=1}^\infty$ is a p-mixing sequence, then $(X, \mathcal{B}, \mu, T)$ is p-mixing.*

3.2 Hilbert Space Preliminaries

3.2.1 Spaces of Hilbert Space-Valued Functions

Whenever we discuss a Hilbert space $\mathcal{H}$, it is naturally endowed with the strong topology. Consequently, the σ-algebra $\mathcal{B}(\mathcal{H})$ of measurable subsets of $\mathcal{H}$ that we want to work with is the completion of the Borel σ-algebra. We will assume here that the reader is familiar with the Bochner integral for vector valued functions (cf. [Yos95] Section V.5 or [Coh13]

Appendix E). Given a measurable space $(X, \mathcal{B})$, a function $f : X \to \mathcal{H}$ is **strongly measurable** if it is a measurable function whose range is separable. We will be working with strongly measurable functions in this chapter since Corollary E.3 of [Coh13] tells us that the set of strongly measurable functions is a vector space, while Exercise E.2 tells us that the set of measurable functions need not be a vector space. Given a probability space $(X, \mathcal{B}, \mu)$ and some $1 \le p \le \infty$, we let $L^p(X, \mu, \mathcal{H})$ denote the collection of strongly measurable functions $f : X \to \mathcal{H}$ for which we also have that $||f||_{\mathcal{H}} \in L^p(X, \mu)$. We will let $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $||\cdot||_{\mathcal{H}}$ denote the inner product and norm of $\mathcal{H}$ respectively so that we may use $\langle \cdot, \cdot \rangle$ and $||\cdot||$ for the norm and inner product of $L^2(X, \mu, \mathcal{H})$. The inner product of $L^2(X, \mu, \mathcal{H})$ is given by $\langle f, g \rangle = \int_X \langle f, g \rangle_{\mathcal{H}} d\mu$. If $(X, \mathcal{B}, \mu, T)$ is a m.p.s., then we let $U : L^2(X, \mu, \mathcal{H}) \to L^2(X, \mu, \mathcal{H})$ be the unitary operator induced by T, which is given by $Uf = f \circ T$. Let $C_{\mathcal{H}}([0,1])$ denote the set of continuous functions $f : [0,1] \to \mathcal{H}$.

Lemma 3.2.1. *Let $([0,1], \mathcal{B}, \mu)$ be a probability space and let $\mathcal{H}$ be a Hilbert space. For each $1 \le p \le \infty$, $C_{\mathcal{H}}([0,1])$ is dense in $L^p([0,1], \mu, \mathcal{H})$.*

Proof. We may use Proposition E.2 and Theorem E.6 of [Coh13] to deduce that f is a norm limit of simple functions in $L^p(X, \mu, \mathcal{H})$. Consequently, it suffices to show that if f is a simple function then f can be approximated arbitrarily closesly by elements of $C_{\mathcal{H}}([0,1])$, and by linearity it suffices to further assume that $f = \xi \mathbb{1}_A$ for some $A \in \mathcal{B}$ and $\xi \in \mathcal{H}$. Let $\epsilon > 0$ be arbitrary. Since any Borel measure μ on $[0,1]$ is regular, let $K \subseteq A$ be a compact set for which $\mu(A \setminus K) < \epsilon$ and let $A \subseteq U$ be an open set for which $\mu(U \setminus A) < \epsilon$. By Urysohn's Lemma let $g : [0,1] \to [0,1]$ be a continuous function for which $g(K) = \{1\}$ and $g([0,1] \setminus U) = \{0\}$. The desired result follows after observing that

$$||f - \xi g||_p = \int_{[0,1]} ||\xi||_{\mathcal{H}} \cdot |\mathbb{1}_A - g| d\mu < ||\xi||_{\mathcal{H}} \mu(U \setminus K) < 2||\xi||_{\mathcal{H}} \epsilon. \tag{3.17}$$

□

Lemma 3.2.2. *Let $([0,1], \mathcal{B}, \mu)$ be a probability space and let $\mathcal{H}$ be a Hilbert space. If $f : [0,1] \to \mathcal{H}$ is a measurable function for which $||f||_{\mathcal{H}} \in L^1(X, \mu)$, then for a.e. $x \in X$ we have that $(f(T^n x))_{n=1}^{\infty} \in UA(\mathcal{H})$.*

Proof. Since $||f(x)||_{\mathcal{H}} \in L^1(X, \mu)$, Birkhoff's Ergodic Theorem shows us that for a.e. $x \in [0,1]$ we have

$$||f||_1 = \int_{[0,1]} ||f(x)||_{\mathcal{H}} d\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} ||f(T^n x)||, \tag{3.18}$$

so $(f(T^n x))_{n=1}^{\infty} \in A(\mathcal{H})$ for a.e. $x \in X$. For each $k \in \mathbb{N}$ let M_k be such that

$$\int_{[0,1]} ||f(x)||_H \mathbb{1}_{||f(x)||_H > M_k}(x) d\mu(x) < \frac{1}{k}. \tag{3.19}$$

Another application of Birkhoff's Ergodic Theorem shows us that

$$\frac{1}{k} > \int_{[0,1]} ||f(x)||_H \mathbb{1}_{||f||_H > M_k}(x) d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} ||f(T^n x)||_H \mathbb{1}_{||f||_H > M_k}(T^n x) \tag{3.20}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} ||f(T^n x)||_H \mathbb{1}_{||f(T^n x)||_H > M_k}(n),$$

so $(f(T^n x))_{n=1}^{\infty} \in UA(H)$ for a.e. $x \in X$. □

3.2.2 Properties of Fully Ergodic and Almost p-Mixing Sequences

Lemma 3.2.3. *Let H be a Hilbert space. If $(x_n)_{n=1}^{\infty} \in UA(H)$ and $(y_n)_{n=1}^{\infty} \in UB(H)$ is such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} ||y_n|| = 0, \text{ then} \tag{3.21}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_n, y_n \rangle = 0. \tag{3.22}$$

Proof. We may assume without loss of generality that $||y_n|| \leq 1$ for all $n \in \mathbb{N}$. Let $(M_q)_{q=1}^{\infty}$ be any sequence for which

$$\lim_{q\to\infty} \frac{1}{M_q} \sum_{n=1}^{M_q} \langle x_n, y_n \rangle \tag{3.23}$$

exists. Let $(N_q)_{q=1}^{\infty}$ be a subsequence of $(M_q)_{q=1}^{\infty}$ for which $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple. Now let $\epsilon > 0$ be arbitrary and let $M > 0$ be such that

$$\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \mathbb{1}_{||x_n|| > M}(n) ||x_n|| < \frac{\epsilon}{2}. \tag{3.24}$$

We now see that

$$\left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_n, y_n \right| \leq \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||x_n||\cdot||y_n|| \tag{3.25}$$

$$\leq \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\mathbb{1}_{||x_n||>M}(n)||x_n||\cdot||y_n|| + \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\mathbb{1}_{||x_n||\leq M}(n)||x_n||\cdot||y_n||$$

$$\leq \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}2\mathbb{1}_{||x_n||>M}(n)||x_n|| + \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}M||y_n|| < \epsilon$$

□

Lemma 3.2.4. *Let H be a Hilbert space and suppose that* $(x_n)_{n=1}^{\infty}$ $UA(H)$.

(i) $(x_n)_{n=1}^{\infty}$ *is fully ergodic if and only if for every invariant* $(y_n)_{n=1}^{\infty}$ *UB(H) we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n, y_n = 0. \tag{3.26}$$

(ii) If $(x_n)_{n=1}^{\infty}$ *is fully ergodic and* $(c_n)_{n=1}^{\infty}$ *UB(C) is invariant, then* $(c_nx_n)_{n=1}^{\infty}$ *is fully ergodic. If H* $= L^2(X,\mu)$ *for a positive* σ*-finite measure* μ*, then we may also allow for* $(c_n)_{n=1}^{\infty}$ *UB(H).*

(iii) Let p_1, p_2 *P(N) be filters such that for every* A_1 p_1 *and* A_2 p_2 *we have* $A_1 \cap A_2 =$ *. If* $(x_n)_{n=1}^{\infty}$ *is almost* p_1*-mixing and* $(y_n)_{n=1}^{\infty}$ *UB(H) is almost* p_2*-rigid then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n, y_n = 0. \tag{3.27}$$

(iv) Let p *P(N) be an idempotent ultrafilter (cf. Page 74).* $(x_n)_{n=1}^{\infty}$ *is almost* p*-mixing if and only if for every almost* p*-rigid* $(y_n)_{n=1}^{\infty}$ *UB(H) we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n, y_n = 0. \tag{3.28}$$

(v) Let p *P(N) be an idempotent ultrafilter. If* $(x_n)_{n=1}^{\infty}$ *is almost* p*-mixing and* $(c_n)_{n=1}^{\infty}$ *UB(C) is almost* p*-rigid, then* $(c_nx_n)_{n=1}^{\infty}$ *is almost* p*-mixing. If H* $= L^2(X,\mu)$ *for a positive* σ*-finite measure* μ*, then we may also allow for* $(c_n)_{n=1}^{\infty}$ *UB(H).*

Proof of (i). For the first direction let us assume that $(x_n)_{n=1}^{\infty}$ is fully ergodic. Lemma 3.2.3 shows us that

$$\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \;\; x_n, y_{n+1} - y_n \;\; = 0. \tag{3.29}$$

Since $(x_n)_{n=1}^{\infty}$ is fully ergodic we see that

$$\begin{aligned} 0 &= \lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \;\; x_{n+h}, y_n \\ &= \lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=h+1}^{N_q} \;\; x_n, y_{n-h} \;\; = \lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=h+1}^{N_q} \;\; x_n, y_n \\ &= \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \;\; x_n, y_n \;\; = \lim_{q\to\infty} \frac{1}{M_q} \sum_{n=1}^{M_q} \;\; x_n, y_n \;\; . \end{aligned} \tag{3.30}$$

For the next direction, let us assume that equation (3.26) holds for every invariant $(y_n)_{n=1}^{\infty}$ $\;\;$ UB(H). For the proof of this direction we will assume familiarity with Chapters 2.2 and 2.3. Let $(z_n)_{n=1}^{\infty}$ $\;\;$ UB(H) be arbitrary, let $(M_q)_{q=1}^{\infty}$ $\;\;$ $\mathbb{N}$ be any sequence for which $((x_n)_{n=1}^{\infty}, (z_n)_{n=1}^{\infty}, (M_q)_{q=1}^{\infty})$ is a weakly permissible triple. Let $\epsilon > 0$ be arbitrary, and let $M > 0$ be such that

$$\lim_{q\to\infty} \frac{1}{M_q} \sum_{n=1}^{M_q} \mathbb{1}_{||x_n||>M}(n)||x_n|| < \frac{\epsilon}{2}. \tag{3.31}$$

Let $(N_q)_{q=1}^{\infty}$ be any subsequence of $(M_q)_{q=1}^{\infty}$ for which $((\mathbb{1}_{||x_n||\le M}(n)x_n)_{n=1}^{\infty}, (z_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple. Letting H denote the induced Hilbert space, let us write $(z_n)_{n=1}^{\infty} = (y_n)_{n=1}^{\infty} + (e_n)_{n=1}^{\infty}$ where $(y_n)_{n=1}^{\infty}$ $\;\;$ UB(H) is invariant and $(e_n)_{n=1}^{\infty}$ $\;\;$ UB(H) is completely ergodic. We may assume without loss of generality that $||y_n||, ||e_n|| \le 1$ for all n. We now see with the aid of the previous direction that

$$\begin{aligned} &\left|\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{M_q} \sum_{n=1}^{M_q} \;\; x_{n+h}, z_n \;\; \right| = \left|\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \;\; x_{n+h}, z_n \;\; \right| \\ =&\left|\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \;\; x_{n+h}, e_n \;\; \right| = \left|\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=h+1}^{N_q} \;\; x_n, e_{n-h} \;\; \right| \\ \le&\, \epsilon + \left|\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=h+1}^{N_q} \;\; \mathbb{1}_{||x_n||\le M}(n)x_n, e_{n-h} \;\; \right| = \epsilon. \end{aligned} \tag{3.32}$$

□

Proof of (ii). If $(y_n)_{n=1}^{\infty} \quad \mathrm{UB}(H)$ is invariant then $(\overline{c_n} y_n)_{n=1}^{\infty}$ is also invariant by Lemma 2.3.8(i). From part (i) we see that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \; c_n x_n, y_n \; = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \; x_n, \overline{c_n} y_n \; = 0, \tag{3.33}$$

so we conclude from part (i) the desired result. The proof when $H = L^2(X,\mu)$ is identical since $L^2(X,\mu)$ has an inner product arising from complex conjugation and multiplication. □

Proof of (iii). We may assume without loss of generality that $||y_n|| \le 1$ for all $n \quad \mathbb{N}$. Let $(M_q)_{q=1}^{\infty}$ be any sequence for which

$$\lim_{q\to\infty} \frac{1}{M_q}\sum_{n=1}^{M_q} \; x_n, y_n \tag{3.34}$$

exists. Let $(N_q)_{q=1}^{\infty}$ be a subsequence of $(M_q)_{q=1}^{\infty}$ for which $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple. Now let $\epsilon > 0$ be arbitrary and let $M > 0$ be such that

$$\lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} \mathbb{1}_{||x_n||>M}(n)||x_n|| < \frac{\epsilon}{2}. \tag{3.35}$$

Since $(x_n)_{n=1}^{\infty}$ is almost p_1-mixing and $(y_n)_{n=1}^{\infty}$ is almost p_2-rigid let $A_1 \quad p_1$ and $A_2 \quad p_2$ be such that

$$\lim_{q\to\infty} \left| \frac{1}{N_q}\sum_{n=1}^{N_q} \; x_{n+h}, y_n \; \right| < \epsilon \text{ for all } h \quad A_1, \tag{3.36}$$

let $M > 0$ be such that

$$\lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} \mathbb{1}_{||x_n||>M}(n)||x_n|| < \epsilon, \tag{3.37}$$

and let

$$\lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} ||y_{n+h} - y_n|| < \frac{\epsilon}{M+1} \text{ for all } h \quad A_2. \tag{3.38}$$

We now see that for $h \quad A_2$ we have

$$\left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_n, y_n - y_{n-h}\right| \le \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||x_n||\cdot||y_n - y_{n-h}|| \tag{3.39}$$

$$\le \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q}\mathbb{1}_{||x_n||>M}(n)||x_n||\cdot||y_n - y_{n-h}||$$

$$+\limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q}\mathbb{1}_{||x_n||\le M}(n)||x_n||\cdot||y_n - y_{n-h}||$$

$$\le \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}2\mathbb{1}_{||x_n||>M}(n)||x_n|| + \limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}M||y_{n+h} - y_n|| < 3\epsilon.$$

We now see that for $h \quad A_1 \cap A_2$ we have

$$\left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_n, y_n\right| = \left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_n, y_n\right| \tag{3.40}$$

$$\le \left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_n, y_n - y_{n-h}\right| + \left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_n, y_{n-h}\right|$$

$$\le \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q}||x_n||\cdot||y_n - y_{n-h}|| + \left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_{n+h}, y_n\right| \le 4\epsilon.$$

□

Proof of (iv). We see that the forwards direction is a consequence of part (iii), so we proceed to prove the backwards direction. For the proof of this direction we will assume familiarity with Chapters 2.2 and 2.3. Let $(z_n)_{n=1}^\infty \quad$ UB(H) be arbitrary, let $(M_q)_{q=1}^\infty \quad \mathbb{N}$ be any sequence for which $((x_n)_{n=1}^\infty, (z_n)_{n=1}^\infty, (M_q)_{q=1}^\infty)$ is a weakly permissible triple. Let $\epsilon > 0$ be arbitrary, and let $M > 0$ be such that

$$\lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q}\mathbb{1}_{||x_n||>M}(n)||x_n|| < \frac{\epsilon}{2}. \tag{3.41}$$

Let $(N_q)_{q=1}^\infty$ be any subsequence of $(M_q)_{q=1}^\infty$ for which $((\mathbb{1}_{||x_n||\le M}(n)x_n)_{n=1}^\infty, (z_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ is a permissible triple. Letting H denote the induced Hilbert space, let us write $(z_n)_{n=1}^\infty = (y_n)_{n=1}^\infty + (e_n)_{n=1}^\infty$ where $(y_n)_{n=1}^\infty \quad$ UB(H) is almost p-rigid (i.e., the projection onto the p-rigid factor) and $(e_n)_{n=1}^\infty \quad$ UB(H) is almost p-mixing. In particular, if $S : H \to H$ is the unitary operator induced by the left shift, then $p - \lim_{n\to\infty} S^n = P$, where convergence takes place in the weak operator topology and P is an orthogonal projection satisfying $p - \lim_{n\to\infty} S^n P = P$. We have that $(y_n)_{n=1}^\infty = P(z_n)_{n=1}^\infty$. We may assume without loss of generality that $||y_n||, ||e_n|| \leq 1$ for all n. We now see with the aid of the previous direction that

$$\left|p-\lim_{h\to\infty}\sum_{h=1}^{H}\lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q} x_{n+h}, z_n\right| = \left|p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_{n+h}, z_n\right| \tag{3.42}$$
$$=p-\left|\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_{n+h}, e_n\right| = p-\left|\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_n, e_{n-h}\right|$$
$$\leq\epsilon+p-\left|\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} \mathbb{1}_{||x_n||\leq M}(n)x_n, e_{n-h}\right| = \epsilon.$$

□

Proof of (v). If $(y_n)_{n=1}^{\infty}$ UB(H) is almost p-rigid then $(\overline{c_n}y_n)_{n=1}^{\infty}$ is seen to also be almost p-rigid as a result of the following calculations.

$$p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||\overline{c_{n+h}}y_{n+h}-\overline{c_n}y_n|| \tag{3.43}$$
$$\leq p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||\overline{c_{n+h}}y_{n+h}-\overline{c_{n+h}}y_n||+p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||\overline{c_{n+h}}y_n-\overline{c_n}y_n||$$
$$\leq p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}M||y_{n+h}-y_n||+p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}M|c_{n+h}-c_n|=0,$$

where M is an upper bound for $|c_n|$ and $||y_n||$. From part (iv) we see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} c_nx_n, y_n = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n, \overline{c_n}y_n = 0, \tag{3.44}$$

so we conclude from part (iv) the desired result. The proof when $H = L^2(X,\mu)$ is identical since $L^2(X,\mu)$ has an inner product arising from complex conjugation and multiplication.

□

Remark 3.2.5. To see why we needed the assumption that $(x_n)_{n=1}^{\infty}$ $UA(H)$ rather than just $(x_n)_{n=1}^{\infty}$ $A(H)$ consider the sequence in $A(\mathbb{C})$ given by

$$x_n = \begin{cases} m & \text{if } n = m^2 \\ -m & \text{if } n = m^2+1 \\ 0 & \text{else} \end{cases}. \tag{3.45}$$

We see that

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|x_n| = 2 < \infty. \tag{3.46}$$

Furthermore, due to the telescoping nature of $(x_n)_{n=1}^{\infty}$ we see that for all $H \in \mathbb{N}$ we have

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\sum_{h=1}^{H}x_{n+h}| = 2 < \infty. \tag{3.47}$$

We now see that if $(y_n)_{n=1}^{\infty} \subseteq UB(\mathbb{C})$ is uniformly bounded in norm by 1 and $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, then

$$\left|\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_{n+h}, y_n \right| = \left|\lim_{H\to\infty}\frac{1}{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\sum_{h=1}^{H}x_{n+h}, y_n \right| \tag{3.48}$$
$$\leq \lim_{H\to\infty}\frac{1}{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}|\sum_{h=1}^{H}x_{n+h}| \leq \lim_{H\to\infty}\frac{2}{H} = 0,$$

so $(x_n)_{n=1}^{\infty}$ is a fully ergodic sequence. However, if we take the invariant sequence $(y_n)_{n=1}^{\infty}$ given by $y_n = 1$ if $n = m^2+1$ and $y_n = -1$ if $n = m^2+1$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n, y_n = 2 = 0. \tag{3.49}$$

This should be compared with the fact that Theorem 2.3.6(i) applies to all elements of $SA(H)$ without any additional uniformity assumptions.

Lemma 3.2.6. *If H is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq UA(H)$ is either a fully ergodic sequence or an almost p-mixing sequence for some filter $p \in P(\mathbb{N})$, then*

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}x_n|| = 0. \tag{3.50}$$

Proof. Let us assume for the sake of contradiction that for some $\epsilon > 0$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ we have

$$\lim_{q\to\infty}||\frac{1}{N_q}\sum_{n=1}^{N_q}x_n|| \geq \epsilon. \tag{3.51}$$

By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ if necessary we may assume without loss of generality that

$$\lim_{q\to\infty}\left(\|\frac{1}{N_q}\sum_{n=1}^{N_{q-1}} x_n\| + \frac{N_{q-1}}{N_q}\right) = 0. \tag{3.52}$$

For $q \in \mathbf{N}$ let

$$\xi_q = \frac{1}{N_q}\sum_{n=N_{q-1}+1}^{N_q} x_n \text{ and } \xi_q = \frac{\xi_q}{\xi_q}. \tag{3.53}$$

Now consider the invariant sequence $(y_n)_{n=1}^{\infty} \in UB(H)$ given by $y_n = \xi_q$ for $N_q \leq n < N_{q+1}$. We see that

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_n, y_n = \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=N_{q-1}+1}^{N_q} x_n, \xi_q = \lim_{q\to\infty}\frac{N_q - N_{q-1}}{N_q}\|\xi_q\| \geq \epsilon, \tag{3.54}$$

which contradicts Lemma 3.2.4 after recalling that any invariant sequence is also an almost p-rigid sequence for any filter $p \in P(\mathbf{N})$. □

The requirement that $(x_n)_{n=1}^{\infty} \in UA(H)$ is an artifact of our method of proof. Lemma 2.3.1(iv) intuitively suggest that we only need $(x_n)_{n=1}^{\infty} \in A(H)$, and we will see that this is indeed the case in Lemma 3.2.7. Even though Lemma 3.2.7 is more general, Lemma 3.2.6 has a more aesthetic proof in the author's opinion. Either of the proofs of Lemma 3.2.6 and 3.2.7 can be used to give an alternative proof of Lemma 2.3.1(iv).

Lemma 3.2.7. *If H is a Hilbert space and $(x_n)_{n=1}^{\infty} \in A(H)$ is either a fully ergodic sequence or an almost p-mixing sequence for some filter $p \in P(\mathbf{N})$, then*

$$\lim_{N\to\infty}\|\frac{1}{N}\sum_{n=1}^{N} x_n\| = 0. \tag{3.55}$$

Proof. We will intuitively proceed in the same fashion as we did in the proof of Lemma 3.2.6, but we will need to use a stronger notion of invariance to overcome the problems illustrated in Remark 3.2.5. More specifically, we begin by showing that if $(y_n)_{n=1}^{\infty} \in UB(H)$ satisfies $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n, y_{n+1} - y_n = 0. \tag{3.56}$$

To this end, it suffices to note that for any $\epsilon > 0$ we have

$$\left|\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} x_n, y_{n+1} - y_n \right| \le \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||x_n||\cdot||y_{n+1}-y_n|| \tag{3.57}$$
$$\le \epsilon \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||x_n||.$$

Let $(y_n)_{n=1}^{\infty} \quad UB(H)$ satisfy $\lim_{n\to\infty}||y_{n+1}-y_n|| = 0$, let $(M_q)_{q=1}^{\infty}$ be any sequence for which

$$\lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q} x_n, y_n \tag{3.58}$$

exists, and let $(N_q)_{q=1}^{\infty}$ be a subsequence of $(M_q)_{q=1}^{\infty}$ for which $((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple. If $(x_n)_{n=1}^{\infty}$ is fully ergodic we see that

$$0 = \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_{n+h}, y_n \tag{3.59}$$
$$= \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_n, y_{n-h} \; = \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_n, y_n$$
$$= \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_n, y_n \; = \lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q} x_n, y_n \; .$$

If $(x_n)_{n=1}^{\infty}$ is almost p-mixing we see that

$$0 = p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_{n+h}, y_n \; = p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=h+1}^{N_q} x_n, y_{n-h} \tag{3.60}$$
$$= p-\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_n, y_n \; = \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q} x_n, y_n \; = \lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q} x_n, y_n \; .$$

Now let us assume for the sake of contradiction that for some $\epsilon > 0$ and $(N_q)_{q=1}^{\infty} \quad \mathbb{N}$ we have

$$\lim_{q\to\infty}||\frac{1}{N_q}\sum_{n=1}^{N_q} x_n|| \ge \epsilon. \tag{3.61}$$

By passing to a subsequence of $(N_q)_{q=1}^{\infty}$ we may assume without loss of generality that

$$\lim_{q\to\infty}\left(||\frac{1}{N_q}\sum_{n=1}^{N_{q-1}+q-1} x_n|| + \frac{N_{q-1}+q-1}{N_q}\right) = 0. \tag{3.62}$$

For q N let

$$\xi_q = \frac{1}{N_q}\sum_{n=N_{q-1}+q}^{N_q} x_n \text{ and } \xi_q = \frac{\xi_q}{\xi_q}. \tag{3.63}$$

Now consider the sequence $(y_n)_{n=1}^{\infty}$ $UB(H)$ given by $y_n = \frac{q-i}{q}\xi_{q-1} + \frac{i}{q}\xi_q$ for $n = N_q + i$ with $0 \leq i < q$ and $y_n = \xi_q$ for $N_q + q \leq n < N_{q+1}$. Since $\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0$ we see that

$$0 = \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=1}^{N_q} x_n, y_n = \lim_{q\to\infty} \frac{1}{N_q}\sum_{n=N_{q-1}+q}^{N_q} x_n, \xi_q = \lim_{q\to\infty} \frac{N_q - N_{q-1}}{N_q}||\xi_q|| \geq \epsilon, \tag{3.64}$$

which yields the desired contradiction. □

3.3 Main Results

Lemma 3.3.1 will allows us to view bounded sequences of Hilbert space vectors as dynamically generated sequences.

Lemma 3.3.1 (cf. Lemma 3.26 of [MRR19]). *Let H be a Hilbert space and let $(a_n)_{n=1}^{\infty}$ $UB(H)$. There exists a compact metric space Y, a continuous map $S : Y \to Y$, a continuous function $F : Y \to H$, and a point y Y with a dense orbit under S such that $a_n = F(S^n(y))$ for all n N.*

Proof. We may assume without loss of generality that $||a_n|| \leq 1$ for all n N. Let B H denote the closed unit ball endowed with the weak topology and let $X = B^{\mathbb{N}}$ endowed with the product topology. Since B is a compact metric space we see that X is also a compact metric space. Let $S : X \to X$ denote the left shift. Let $F : X \to H$ denote the projection onto the first coordinate, which is seen to be continuous. Let $x = (a_n)_{n=1}^{\infty}$ X, and let $X = c\ell(\{S^n x\}_{n=1}^{\infty})$. Since X is a closed subset of X we see that X is also a compact metric space, and it is clear that x has a dense orbit in X by construction. □

Lemma 3.3.3 allows us to convert correlations of sequences to inner products of functions in a Hilbert space, which will then permit us to use Hilbert space Theory to analyze our correlations. Before stating Lemma 3.3.3, let us recall the definition of a generic point.

Definition 3.3.2. *Given an ergodic m.p.s. $([0,1], B, \mu, T)$, x $[0,1]$ is* ***generic*** *if for every continuous function $f : [0,1] \to \mathbb{C}$ we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^n x) = \int_{[0,1]} f d\mu. \tag{3.65}$$

A well known consequence of Birkho 's Ergodic Theorem is that a.e. x X is generic.

Lemma 3.3.3. *Let* $([0,1], \mathcal{B}, \mu, T)$ *be an ergodic m.p.s., let* $x \quad [0,1]$ *be generic, and let* $\mathcal{H}$ *be a Hilbert space. Let* $(y_n)_{n=1}^{\infty} \quad UB(\mathcal{H})$ *and let* $f : C_{\mathcal{H}}([0,1])$ *both be arbitrary. Let* $U : L^2([0,1], \mu, \mathcal{H}) \to L^2([0,1], \mu, \mathcal{H})$ *be the unitary operator induced by* T*. Let* $(N_q)_{q=1}^{\infty} \quad \mathbb{N}$ *be any sequence for which*

$$\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f(T^{n+h}x), y_n \ _{\mathcal{H}} \tag{3.66}$$

exists for each $h \quad \mathbb{N}$*. Then there exists* $g \quad L^2([0,1], \mu, \mathcal{H})$ *for which*

$$\lim_{q \to \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} f(T^{n+h}x)\overline{y_n} = \ U^h f, g \ _{\mu}. \tag{3.67}$$

Proof. By Lemma 3.3.1, let Y be a compact metric space, let $F \quad C(Y)$, $y \quad Y$, and $S : Y \to Y$ be such that $y_n = F(S^n y)$. Let ν be any weak limit point of the sequence

$$\left(\frac{1}{N_q} \sum_{k=1}^{N_q} \delta_{T^n x, S^n y}\right)_{q=1}^{\infty}, \tag{3.68}$$

and let $(M_q)_{q=1}^{\infty} \quad \mathbb{N}$ be such that

$$\nu = \lim_{q \to \infty} \frac{1}{M_q} \sum_{n=1}^{M_q} \delta_{T^n x, S^n y}. \tag{3.69}$$

Let $\tilde{f}, \tilde{F} \quad L^2([0,1] \times Y, \nu, \mathcal{H})$ be given by $\tilde{f}(x,y) = f(x)$ and $\tilde{F}(x,y) = F(y)$. Let $V : L^2([0,1] \times Y, \nu, \mathcal{H}) \to L^2([0,1] \times Y, \nu, \mathcal{H})$ be the unitary operator induced by $T \times S$. We note that if $h \quad L^2([0,1] \times Y, \nu, \mathcal{H})$ is a continuous function for which $h(x,y) = k(x)$ for some $k \quad L^2([0,1], \mu, \mathcal{H})$, then the genericity of x shows us that for any continuous functional $\rho : \mathcal{H} \to \mathbb{C}$ we have

$$\rho(\int_{[0,1] \times Y} h d\nu) = \int_{[0,1] \times Y} \rho(h) d\nu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \rho(h(T^n x, S^n y)) \tag{3.70}$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \rho(k(T^n x)) = \int_{[0,1]} \rho(k) d\mu = \rho(\int_{[0,1]} k d\mu), \text{ hence}$$

$$\int_{[0,1] \times Y} h d\nu = \int_{[[0,1]} k d\mu. \tag{3.71}$$

Since Lemma 3.2.1 tells us that the continuous functions are dense in $L^2(X, \mu, \mathcal{H})$, we see that equation (3.71) holds even when h and k are not continuous. Let $\tilde{\mu}$ be the probability measure on $([0,1]\times Y, \mathcal{B}\otimes\mathcal{A})$ given by $\tilde{\mu}(B\times A)=\mu(B)\mathbb{1}_A(y)$ for all $B\in\mathcal{B}$ and $A\in\mathcal{A}$. Since we may identify $L^2([0,1]\times Y,\tilde{\mu})$ with the functions in $L^2([0,1]\times Y,\nu,\mathcal{H})$ of the form $h(x,y)=k(x)$, let $P: L^2([0,1]\times Y,\nu,\mathcal{H})\to L^2([0,1]\times Y,\tilde{\mu},\mathcal{H})$ denote the orthogonal projection. Let $\tilde{g}=P\tilde{\xi}$, and let $g\in L^2([0,1],\mu)$ be such that $\tilde{g}(x,y)=g(x)$. Since f and F are continuous functions we see that $\langle\tilde{f}(T^h\cdot),\tilde{F}(\cdot)\rangle:[0,1]\times Y\to\mathbb{C}$ is a continuous function, so for all $h\in\mathbb{N}$ we have

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f(T^{n+h}x),y_n\rangle_{\mathcal{H}}=\lim_{q\to\infty}\frac{1}{M_q}\sum_{n=1}^{M_q}\langle\tilde{f}(T^{n+h}x),\tilde{F}(S^ny)\rangle_{\mathcal{H}} \tag{3.72}$$
$$=\int_{[0,1]\times Y}\langle V^hf,\tilde{F}\rangle_{\mathcal{H}}d\nu=\langle V^h\tilde{f},\tilde{F}\rangle_\nu=\langle V^h\tilde{f},\tilde{g}\rangle_\nu=\langle U^hf,g\rangle_\mu.$$

□

Lemma 3.3.4. *Let $([0,1],\mathcal{B},\mu,T)$ be an ergodic m.p.s. and let $\mathcal{H}$ be a Hilbert space. For each generic point $x\in[0,1]$ and each $f\in C_{\mathcal{H}}([0,1])$ with $\int_{[0,1]}fd\mu=0$, $(f(T^nx))_{n=1}^\infty$ is a fully ergodic sequence.*

Proof. Let $f\in C_{\mathcal{H}}([0,1])$ and $(y_n)_{n=1}^\infty\in UB(\mathcal{H})$ both be arbitrary. Let $(N_q)_{q=1}^\infty\subseteq\mathbb{N}$ be such that $((f(T^nx))_{n=1}^\infty,(y_n)_{n=1}^\infty,(N_q)_{q=1}^\infty)$ is a weakly permissible triple. By Lemma 3.3.3, let $g\in L^2([0,1],\mu,\mathcal{H})$ be such that

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f(T^{n+h}x),y_n\rangle_{\mathcal{H}}=\langle U^hf,g\rangle_\mu. \tag{3.73}$$

Letting $U:L^2([0,1],\mu,\mathcal{H})\to L^2([0,1],\mu,\mathcal{H})$ denote the unitary operator induced by T, we see that U is ergodic. It follows that

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f(T^{n+h}x),y_n\rangle_{\mathcal{H}}=\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\langle U^hf,g\rangle_\mu=0. \tag{3.74}$$

□

Theorem 3.3.5. *Let $([0,1],\mathcal{B},\mu,T)$ be an ergodic m.p.s., let $\mathcal{H}$ be a Hilbert, and let $f\in L^1([0,1],\mu,\mathcal{H})$ satisfy $\int_{[0,1]}fd\mu=0$. For a.e. $x\in[0,1]$, $(f(T^nx))_{n=1}^\infty$ is a fully ergodic sequence.*

Proof. Let $\epsilon>0$ be arbitrary, and by Lemma 3.2.1 let $g\in C_{\mathcal{H}}([0,1])$ satisfy $||f-g||_1\le\epsilon$ and $\int_{[0,1]}gd\mu=0$. Let $X\subseteq[0,1]$ be a set of full measure for which Birkhoff's Ergodic Theorem holds for the real-valued function $||f-g||_{\mathcal{H}}$ along every $x\in X$. We now see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}||f(T^nx)-g(T^nx)||_H=\int_{[0,1]}||f-g||_H d\mu=||f-g||_1<\epsilon. \tag{3.75}$$

Now let $(y_n)_{n=1}^{\infty}$ $UB(H)$ be uniformly bounded in norm by 1. Since a.e. x X is a generic point we may use Lemma 3.3.4 to further refine X to another set of full measure X , such that for every x X , $(g(T^nx))_{n=1}^{\infty}$ is a fully ergodic sequence. We see that for any x X and any permissible triple $(f(T^nx)_{n=1}^{\infty},(y_n)_{n=1}^{\infty},(N_q)_{q=1}^{\infty})$, we have

$$\left|\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}|\frac{1}{N_q}\sum_{n=1}^{N_q}f(T^{n+h}x),y_n\right| \tag{3.76}$$
$$\leq\left|\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\lim_{q\to\infty}|\frac{1}{N_q}\sum_{n=1}^{N_q}g(T^{n+h}x),y_n\right|+\epsilon=\epsilon.$$

Since $\epsilon>0$ was arbitrary, we see that $(f(T^nx))_{n=1}^{\infty}$ is a fully ergodic sequence. □

Corollary 3.3.6. *Let* $([0,1],B,\mu,T)$ *be an ergodic m.p.s., let* H *be a Hilbert space, and let* f $L^1([0,1],\mu,H)$. *For a.e.* x $[0,1]$ *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^nx)=\int_{[0,1]}fd\mu, \tag{3.77}$$

with convergence taking place in the strong topology. Furthermore, if $\int_{[0,1]}fd\mu=0$*, then for a.e.* x $[0,1]$ *we have*

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}c_nf(T^nx)||=0, \tag{3.78}$$

where $(c_n)_{n=1}^{\infty}$ $UB(\mathbb{C})$ *is any invariant sequence.*

Proof. We see that $f := f-\int_{[0,1]}fd\mu$ satisfies $\int_{[0,1]}f\,d\mu=0$, so by Theorem 3.3.5 we see that $(f\,(T^nx))_{n=1}^{\infty}$ is a fully ergodic sequence for a.e. x X. By Lemma 3.2.2 we see that $(f\,(T^nx))_{n=1}^{\infty}$ $UA(H)$. By Lemma 3.2.6 we see that for a.e. x X we have

$$0=\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}f\,(T^nx)||=\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}f(T^nx)-\int_{[0,1]}fd\mu||. \tag{3.79}$$

The latter half of the Theorem is a consequence of Lemmas 3.3.4 and 3.2.4(ii). □

We observe that the first half of Corollary 3.3.6 can also be derived as a consequence of the main result of [Cha62] for $f \in L^p([0,1],\mu,H)$ if $p > 1$ (see also [HST78]).

Lemma 3.3.7. *Let $([0,1],\mathcal{B},\mu,T)$ be an ergodic m.p.s., let H be a Hilbert space, and let $p \in \mathcal{P}(\mathbb{N})$ be a filter. For each generic point $x \in [0,1]$ and each continuous function $f:[0,1]\rightarrow H$ with $\int_{[0,1]} f d\mu = 0$, $(f(T^n x))_{n=1}^{\infty}$ is a p-mixing sequence.*

Proof. Let $f \in C_H([0,1])$ and $(y_n)_{n=1}^{\infty} \in UB(H)$ both be arbitrary. Let $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ be such that $((f(T^n x))_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple. By Lemma 3.3.3, let $g \in L^2([0,1],\mu,H)$ be such that

$$\lim_{q\rightarrow\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f(T^{n+h}x), y_n\rangle_H = \langle U^h f, g\rangle_\mu. \tag{3.80}$$

Letting $U: L^2([0,1],\mu,H)\rightarrow L^2([0,1],\mu,H)$ denote the unitary operator induced by T, we see that U is p-mixing. It follows that

$$p-\lim_{h\rightarrow\infty}\lim_{q\rightarrow\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f(T^{n+h}x), y_n\rangle_H = p-\lim_{h\rightarrow\infty}\langle U^h f, g\rangle_\mu = 0. \tag{3.81}$$

□

Theorem 3.3.8. *Let $p \in \mathcal{P}(\mathbb{N})$ be a filter and let $([0,1],\mathcal{B},\mu,T)$ be an p-mixing m.p.s., let H be a Hilbert space, and let $f \in L^1([0,1],\mu,H)$ satisfy $\int_{[0,1]} f d\mu = 0$. For a.e. $x \in [0,1]$, $(f(T^n x))_{n=1}^{\infty}$ is a p-mixing sequence.*

Proof. Let $\epsilon > 0$ be arbitrary, and by Lemma 3.2.1 let $g \in C_H([0,1])$ satisfy $||f-g||_1 \le \epsilon$ and $\int_{[0,1]} g d\mu = 0$. Let $X \subseteq [0,1]$ be a set of full measure for which Birkhoff's Ergodic Theorem holds for the real-valued function $||f-g||_H$ along every $x \in X$. We now see that

$$\lim_{N\rightarrow\infty}\frac{1}{N}\sum_{n=1}^{N}||f(T^n x)-g(T^n x)||_H = \int_{[0,1]}||f-g||_H d\mu = ||f-g||_1 < \epsilon. \tag{3.82}$$

Now let $(y_n)_{n=1}^{\infty} \in UB(H)$ be uniformly bounded in norm by 1. Since a.e. $x \in X$ is a generic point we may use Lemma 3.3.7 to further refine X to another set of full measure X', such that for every $x \in X'$, $(g(T^n x))_{n=1}^{\infty}$ is a p-mixing sequence. We see that for any $x \in X'$ and any permissible triple $((f(T^n x))_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, we have

$$\left|p-\lim_{h\to\infty}\lim_{q\to\infty}\langle\frac{1}{N_q}\sum_{n=1}^{N_q}f(T^{n+h}x),y_n\rangle\right| \tag{3.83}$$
$$\leq\left|p-\lim_{h\to\infty}\lim_{q\to\infty}\langle\frac{1}{N_q}\sum_{n=1}^{N_q}g(T^{n+h}x),y_n\rangle\right|+\epsilon=\epsilon.$$

Since $\epsilon>0$ was arbitrary, we see that $(f(T^nx))_{n=1}^{\infty}$ is a almost p-mixing sequence. □

We state our next corollary in the language of probability since the Strong Law of Large Numbers is an ubiquitous result in probability that is analogous to Birkhoff's Ergodic Theorem. We recall that p_c $\mathcal{P}(\mathbb{N})$ denote the cofinite filter.

Corollary 3.3.9. *Let H be a Hilbert space and let $(X_n)_{n=1}^{\infty}$ be an i.i.d. sequence of strongly measurable H-valued random variables with $\mathbb{E}[X_i]=0$. $(X_n)_{n=1}^{\infty}$ is almost surely an almost p_c-mixing sequence. In particular, we almost surely have*

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}c_nX_n||=0, \tag{3.84}$$

where $(c_n)_{n=1}^{\infty}$ $UB(\mathbb{C})$ is any rigid sequence.[1]

Proof. Let $(\Omega,\mathcal{F},\mathbb{P})$ denote the underlying probability space. Since $(X_n)_{n=1}^{\infty}$ is i.i.d., we may assume that there exists a probability preserving transformation $T:\Omega\to\Omega$ that is Bernoulli for which we also have $X_n=X_1\circ T^n$. Since any Bernoulli system is also strongly mixing we may apply Theorem 3.3.8 to see that $(X_n)_{n=1}^{\infty}$ is almost surely a p_c-mixing system. The astute reader may have observed that we need not have $(\Omega,\mathcal{F})=([0,1],\mathcal{B})$, but Theorem 3.3.14 shows that this is not an issue. The latter part of the corollary follows from Lemmas 3.2.4(v) and 3.2.6. □

Our next result shows that Theorem 3.3.8 characterizes almost p-mixing systems.

Theorem 3.3.10. *Let p $\mathcal{P}(\mathbb{N})$ be a filter and $(X,\mathcal{B},\mu,T)$ a m.p.s. If for every f $L^{\infty}(X,\mu)$ with $\int_X fd\mu=0$ there exists a set A_f $\mathcal{B}$ satisfying $\mu(A_f)=1$ and for every x A_f the sequence $(f(T^nx))_{n=1}^{\infty}$ is almost p-mixing, then $(X,\mathcal{B},\mu,T))$ is p-mixing.*

Proof. Let A,B $\mathcal{B}$ both be arbitrary. Let X X be a set of full measure, such that for any x X and all h $\mathbb{N}$ we have that $(\mathbb{1}_B(T^nx)-\mu(B))_{n=1}^{\infty}$, $(\mathbb{1}_A(T^nx)-\mu(A))_{n=1}^{\infty}$

[1]Here we are referring to rigid sequences as in Definition 2.3.2, which is equivalent to $(c_n)_{n=1}^{\infty}$ being almost p-rigid for some filter p. We also remark that $(c_n)_{n=1}^{\infty}$ is almost p-rigid for some filter p if and only if it is almost q-rigid for some idempotent ultrafilter q.

and $\left(\mathbb{1}_{T^{-h}(A)\cap B}(T^n x) - \mu(T^{-h}(A)\cap B)\right)_{n=1}^{\infty}$ are p-mixing sequences. Lemma 3.2.6 shows us that for any x X, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_B(T^n x) = \mu(B) \text{ and} \tag{3.85}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{T^{-h}(A)\cap B}(T^n x) = \mu(T^{-h}(A)\cap B) \tag{3.86}$$

for every h N. We now see that for any x X, we have

$$p-\lim_{h\to\infty}\mu(T^{-h}(A)\cap B) = p-\lim_{h\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{T^{-h}(A)\cap B}(T^n x) \tag{3.87}$$

$$=p-\lim_{h\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{T^{-h}(A)}(T^n x)\mathbb{1}_B(T^n x)$$

$$=p-\lim_{h\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_A(T^{n+h} x)\mathbb{1}_B(T^n x)$$

$$=p-\lim_{h\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(\mathbb{1}_A(T^{n+h} x)-\mu(A))\mathbb{1}_B(T^n x)$$

$$+p-\lim_{h\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A)\mathbb{1}_B(T^n x)$$

$$=0+\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A)\mathbb{1}_B(T^n x) = \mu(A)\mu(B).$$

□

Now let us consider Theorem 3.3.11, which is a corollary of Proposition 3.1.11.

Theorem 3.3.11. *Let* p *P*(N) *be a filter and* $(X, \mathcal{B}, \mu, T)$ *a m.p.s. If for every* f $L^\infty(X,\mu)$ *with* $\int_X f d\mu = 0$ *there exists a set* A_f $\mathcal{B}$ *satisfying* $\mu(A_f) = 1$ *and for every* x A_f *the sequence* $(f(T^n x))_{n=1}^{\infty}$ *is fully ergodic, then* $(X, \mathcal{B}, \mu, T))$ *is ergodic.*

It is natural to ask if Theorem 3.3.11 can be improved by reducing the class of test functions f to a proper subset of $L^\infty(X,\mu)$ since we are using a stronger assumption on the pointwise orbits of f than we did in Proposition 3.1.11. A natural candidate in the case of $X = [0,1]$ would be $C([0,1])$. Our next theorem shows that this is not the case even if we further strengthen the assumptions on the pointwise orbits of f.

Theorem 3.3.12. *Let* p_c *P*(N) *be the cofinite filter. There exists a non-ergodic m.p.s.* $([0,1], \mathcal{B}, m, T)$ *such that for a.e.* x X *and every* f $C([0,1])$ *the sequence* $(f(T^n x))_{n=1}^{\infty}$ *is almost* p_c*-mixing.*

Proof. Let m be Lebesgue measure. Let $S : [0,1] \to [0,1]$ be any strongly mixing transformation. Let $I_1 \in \mathcal{B}$ be such that $0 < m(I_1 \cap U) < m(U)$ for any open interval U (cf. [Kir72]). Let $I_2 = I_1^c$, and note that $0 < m(I_2) < 1$ and $m(I_2 \cap U) > 0$ for any open interval U. We will now show that for $j \in \{1,2\}$ there exists measures $\nu_j : \mathcal{B} \to [0,1]$ supported on I_j for which $\nu_1(I_j \cap U) := m(U)$ for every open interval U. Firstly, we note that if $U_1, U_2 \subseteq [0,1]$ are open intervals for which $m(U_1) = m(U_2)$, then there exists nonempty open intervals U_3, U_4, U_5 for which $U_1 = U_3 \cup U_4$ and $U_2 = U_4 \cup U_5$ up to one missing point. Furthermore, we see that $m(U_3) = m(U_5)$, so

$$\nu_j(I_j \cap U_1) = \nu_j(I_j \cap U_3) + \nu_j(I_j \cap U_4) = \nu_j(I_j \cap U_5) + \nu_j(I_j \cap U_4) = \nu_j(I_j \cap U_2), \quad (3.88)$$

so each ν_j is a well defined map. Furthermore, we see that the maps ν_j induce outer measures ν_j^* by

$$\nu_j^*(E) = \inf\{\sum_{n=1}^{\infty} \nu_j(E \cap U_n) \mid \{U_n\}_{n=1}^{\infty} \text{ is a cover of } E \text{ by open intervals}\}. \quad (3.89)$$

Since each ν_j^* is a Carathéodory outer measure, we see that each ν_j is a well defined measure on $\mathcal{B}$ (cf. Section 12.8 of [Roy88]).

By Theorem 15.5.16 of [Roy88] let $\phi_j : I_j \to [0,1]$ be a measurable isomorphism satisfying $\nu_j = \phi_* m$. Let $T : [0,1] \to [0,1]$ be given by

$$T(x) = \begin{cases} \phi_1^{-1}(S(\phi_1(x))) & \text{if } x \in I_1 \\ \phi_2^{-1}(S(\phi_2(x))) & \text{if } x \in I_2 \end{cases}. \quad (3.90)$$

Since $m(I_1) \notin \{0,1\}$ and $T(I_1) = I_1$, we see that T is not ergodic. Now let $f \in C([0,1])$ satisfy $\int_0^1 f(x)dm(x) = 0$ and note that $f \in L^1([0,1],\nu_1) \cap L^1([0,1],\nu_2)$. We see from the construction of ν_1 and ν_2 that

$$\int_{[0,1]} f d\nu_1 = \int_{[0,1]} f d\nu_2 = \int_{[0,1]} f dm = 0, \quad (3.91)$$

so Theorem 3.3.8 shows us that for a.e. $x \in I_1 \cup I_2 = [0,1]$ the sequence $(f(T^n x))_{n=1}^{\infty}$ is almost p_c-mixing. □

Remark 3.3.13. We observe that in the proof above we could have taken S to be a Bernoulli transformation, but we chose not to do so since we did not discuss any properties of the pointwise orbits of $f \in L^1(X,\mu)$ for a Bernoulli m.p.s. $(X, \mathcal{B}, \mu, T)$. We will conclude this section by deducing that Theorem 3.3.5 and Theorem 3.3.8 hold for arbitrary measure preserving systems as a consequence of Theorem 3.3.14. For the proof of Theorem 3.3.14 we will be using vocabulary and notation from chapter 15 of [Roy88] that we will not discuss here.

Theorem 3.3.14. *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s., let $\mathcal{H}$ be a Hilbert space, and let $f \in L^1(X, \mu, \mathcal{H})$. There exists a m.p.s. $([0,1], \mathcal{B}, m, S)$ and $\tilde{f} \in L^1([0,1], m, \mathcal{H})$ such that for a.e. $x \in X$, there exists $y = y(x)$ satisfying $f(T^n x) = \tilde{f}(S^n x)$.*

Proof. Since the range of f is separable, let $\{U_k\}_{k=1}^{\infty}$ be a basis for subspace topology on closure of the range of f. Let $\mathcal{A}$ denote the σ-algebra generated by $\{\{f^{-1}(U_k)\}_{k=1}^{\infty}\}$. We see that $\mathcal{A}$ is a countably generated σ-algebra with respect to which f is measurable. By Carathéodory's Theorem (cf. Theorem 15.3.4 of [Roy88]) there exists an isomorphism Φ of $< \mathcal{A}, \mu >$ into $< \mathcal{L}/\mathcal{N}, m >$, where $\mathcal{L}$ is the Lebesgue σ-algebra on $[0,1]$, m is Lebesgue measure and $\mathcal{N} \subset \mathcal{L}$ is the σ-algebra of null sets. By Proposition 15.6.19 of [Roy88], let $\phi : X \to [0,1]$ be a measurable transformation for which $\mu(\phi^{-1}(B) \triangle \Phi^{-1}(B)) = 0$ for every $B \in \Phi(\mathcal{A})$.

By Proposition E.2 of [Coh13] let $\{f_n\}_{n=1}^{\infty}$ be a sequence of simple functions that converge pointwise a.e. to f. For each $n \in \mathbb{N}$ let $\{\xi_{n,k}\}_{k=1}^{K_n}$ denote the range of f_n, and let $A_{n,k} = f_{n,k}^{-1}(\{\xi_{n,k}\})$. We may now define $\tilde{f}_n = \sum_{k=1}^{K_n} \xi_{n,k} \mathbb{1}_{\Phi(A_{n,k})}$ and observe that $\tilde{f}_n \circ \phi = f_n$ a.e. for all $n \in \mathbb{N}$. Since f_n converges pointwise a.e. to $f \in L^1(X, \mu, \mathcal{H})$, we see that $\tilde{f}_n$ converges pointwise a.e. to some $\tilde{f} \in L^1([0,1], m, \mathcal{H})$ satisfying $\tilde{f} \circ \phi = f$ a.e. Now note that $\Phi \circ T^{-1} \circ \Phi^{-1}$ is a σ-isomorphism of $\Phi(\mathcal{A})$ to itself, so another application of Proposition 15.6.19 of [Roy88] yields a map $S : [0,1] \to [0,1]$ for which $S^{-1}(B) = \Phi(T^{-1}(\Phi^{-1}(B)))$ for every $B \in \Phi(\mathcal{A})$. Noting that $\phi^{-1} \circ S^{-1}$ and $T^{-1} \circ \phi^{-1}$ are the same σ-homomorphism, we can use the uniqueness portion of Proposition 15.6.9 of [Roy88] to see that $S \circ \phi = \phi \circ T$. It follows that for some $X' \subset X$ with $\mu(X') = 1$ we have that $S^n(\phi(x)) = \phi(T^n(x))$ and $\tilde{f}(S^n(\phi(x))) = f(T^n(x))$ for all $n \in \mathbb{N}$ and all $x \in X'$, so it suffices to take $y = y(x) = \phi(x)$. □

3.4 Conjectures and Questions

One way in which we can try to generalize Theorem 3.3.8 when $p = p_D$ is the filter of sets of natural density 1, is motivated by the following Theorem of Bourgain.

Theorem 3.4.1 (Theorem 1 in [Bou89]). *Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s. and let $p(x)$ be a polynomial with integer coefficients. If $f \in L^r(X, \mu)$ for some $r > 1$, then*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{p(n)} x) \tag{3.92}$$

exists for a.e. $x \in X$. Furthermore, if T is weakly mixing, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^{p(n)}x)=\int_X f d\mu \tag{3.93}$$

for a.e. $x \in X$.

Theorem 3.4.1 shows us that Birkhoff's Ergodic Theorem holds along polynomial subsequences if the underlying transformation T is assumed to be weakly mixing. This naturally leads us to ask if some form of Theorem 3.3.5 holds for polynomial subsequences.

Question 3.4.2. *If $(X, \mathcal{B}, \mu, T)$ is a weakly mixing m.p.s., $p(x)$ a polynomial with integer coefficients and $f \in L^r(X,\mu)$ with $r > 1$ is such that $\int_X f d\mu = 0$, then is $(f(T^{p(n)}x))_{n=1}^{\infty}$ an almost p_D-mixing sequence for a.e. $x \in X$? What can be said for other levels of mixing? What can be said for elements of $L^r(X,\mu,\mathcal{H})$?*

We see that if $(x_n)_{n=1}^{\infty}$ is an almost p-mixing sequence of complex numbers, and $(y_n)_{n=1}^{\infty}$ is another sequence of complex numbers for which $d(\{n \in \mathbb{N} \mid x_n = y_n\}) = 0$, then $(y_n)_{n=1}^{\infty}$ is also p-mixing. In particular, if $(x_n)_{n=1}^{\infty}$ is p_D-mixing, and $(y_n)_{n=1}^{\infty}$ is given by $y_n = x_n$ when n is not a square and $y_n = 1$ when n is a square, then $(y_n)_{n=1}^{\infty}$ is also a p_D-mixing sequence, but $(y_{n^2})_{n=1}^{\infty}$ is the constant sequence, so Question 3.4.2 does not follow immediately from Theorem 3.3.8. It is well known (cf. [Ber87]) that if $(X, \mathcal{B}, \mu, T)$ is a weakly mixing system, then for any $f, g \in L^2(X,\mu)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left|\langle U^{n^2}f, g\rangle - \int_X f d\mu\int_X g d\mu\right| = 0. \tag{3.94}$$

Combining this fact with Lemma 3.3.3 allows us to prove Proposition 3.4.3, but does not immediately help us resolve Question 3.4.2.

Proposition 3.4.3. *Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing m.p.s., let $\mathcal{H}$ be a Hilbert space, and let $f \in L^1(X,\mu,\mathcal{H})$ satisfy $\int_X f d\mu = 0$. For a.e. $x \in X$, any $(y_n)_{n=1}^{\infty} \in UB(\mathcal{H})$, and any $(N_q)_{q=1}^{\infty}$ for which $((f(T^n x))_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a weakly permissible triple, we have*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f(T^{n+h^2}x), y_n\rangle\right| = 0. \tag{3.95}$$

We also see that Lemma 3.3.3 suggests we do not need the m.p.s. $([0,1], \mathcal{B}, \mu, T)$ in Theorem 3.3.5 to be weakly mixing. In particular, if we work with $L^2([0,1],\mu)$ instead of $L^1([0,1],\mu)$, then it seems that we only need $f \in L^2([0,1],\mu)$ to satisfy

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}| \ U^n f, g \ - \int_X f d\mu \int_X g d\mu| = 0. \tag{3.96}$$

for every $g \quad L^2([0,1],\mu)$. However, it is not obvious as to whether or not this is the case. In the proofs of Theorems 3.3.5 and 3.3.8, we approximated f by continuous functions, and every element of $C([0,1]) \quad L^2([0,1],\mu)$ satisfying equation 3.96 is the same as the m.p.s. $([0,1],\mathcal{B},\mu,T)$ being weakly mixing. This leads us to Conjecture 3.4.4.

Conjecture 3.4.4. *Let $(X,\mathcal{B},\mu,T)$ be a m.p.s., let $\mathcal{H}$ be a Hilbert space, and let $p \quad \mathcal{P}(\mathbb{N})$ be a filter, and let $f \quad L^2(X,\mu,\mathcal{H})$ satisfy*

$$p - \lim_{n\to\infty} \ U^n f, g \ = 0 \tag{3.97}$$

for every $g \quad L^2(X,\mu)$. Then $(f(T^n x))_{n=1}^{\infty}$ is an almost p-mixing sequence for a.e. $x \quad X$.

In light of Chapter 2, we observe that if $f \quad L^2(X,\mu,\mathcal{H})$, then $(f(T^n x))_{n=1}^{\infty} \quad SA(\mathcal{H})$ for a.e. $x \quad X$. It is natural to ask if we further assume that $(X,\mathcal{B},\mu,T)$ is ergodic if $(f(T^n x))_{n=1}^{\infty}$ is a completely ergodic sequence, but Remark 2.3.7 shows us that it is not obvious whether or not a fully ergodic sequence in $SA(\mathcal{H})$ is also a completely ergodic sequence. This leads us to our last question for this chapter.

Question 3.4.5. *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \quad SA(\mathcal{H})$ is a fully ergodic sequence, is it also a completely ergodic sequence? What if we assume that $(x_n)_{n=1}^{\infty}$ is given by $x_n = f(T^n x)$ for some ergodic $(X,\mathcal{B},\mu,T)$ and $f \quad L^2(X,\mu,\mathcal{H})$? What can be said about other levels of mixing?*

CHAPTER 4

UNIFORM SYMMETRIC DISTRIBUTION

4.1 Introduction

A sequence $(r_n)_{n=1}^{\infty}$ Z is an **ergodic sequence** if for any α R the sequence $(r_n\alpha)_{n=1}^{\infty}$ (mod 1) is uniformly distributed within its orbit closure. Such sequences are named ergodic sequences because it is well known that the Mean Ergodic Theorem holds along such subsequences. To be more precise, we have the following Theorem as a consequence of the main result of [BE74].

Theorem 4.1.1. *For $R = (r_n)_{n=1}^{\infty}$ Z the following are equivalent.*

(i) R is an ergodic sequence.

(ii) If H is a Hilbert space, and $U : H \to H$ is a unitary operator, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} U^{r_n} = P, \tag{4.1}$$

where convergence takes place in the strong operator topology, and $P : H \to H$ is the projection onto the largest U-invariant subspace of H.

(iii) For any ergodic invertible m.p.s. (X, B, μ, T), and any A, B B, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{r_n}B) = \mu(A)\mu(B). \tag{4.2}$$

It is natural to wonder if Theorem 4.1.1(iv) has an equivalent symmetric form using a single A B. To better understand what is meant, let us consider for example the following well known result about equivalent characterizations of ergodic measure preserving systems.

Theorem 4.1.2 (cf. Theorem 1.23 in [Wal75]). *Let (X, B, μ, T) be an invertible m.p.s. We have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^nA)=\mu(A)^2 \tag{4.3}$$

for all $A \in \mathcal{B}$ if and only if we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^nB)=\mu(A)\mu(B) \tag{4.4}$$

for all $A, B \in \mathcal{B}$.

In particular, the principal of various levels of the ergodic hierarchy of mixing having equivalent characterizations of a symmetric nature is the basis of Chapter 2. We say that $(r_n)_{n=1}^{\infty} \subseteq \mathbb{Z}$ is a **semi-ergodic sequence** if for any ergodic invertible m.p.s. $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^nA)=\mu(A)^2. \tag{4.5}$$

In light of Theorem 4.1.2 and Chapter 2 we would hope that semi-ergodic sequences and ergodic sequences are the same, but we will show in this chapter that this is not the case. Notably, we have the following theorem as one of the main results of this chapter.

Theorem 4.1.3 (cf. Theorem 4.2.1)**.** *For $(r_n)_{n=1}^{\infty} \subseteq \mathbb{Z}$ conditions (i)-(iv) are equivalent.*

(i) (A) For every irrational $\alpha \in \mathbb{T}$, and every $U \subseteq \mathbb{T}$ that is a finite union of open intervals for which $U = -U$, we have

$$\lim_{N\to\infty}\frac{1}{N}|\{1\le n\le N \mid r_n\alpha \in U\}|=\mu(U). \tag{4.6}$$

(B) For every integer $q \ge 1$, we have

(a) $\displaystyle\lim_{N\to\infty}\frac{|(r_n)_{n=1}^{N}\cap q\mathbb{N}|}{N}=\frac{1}{q}$,

(b) $\displaystyle\lim_{N\to\infty}\frac{|(r_n)_{n=1}^{N}\cap((q\mathbb{N}+j)\cup(q\mathbb{N}-j))|}{N}=\frac{2}{q}$*, and*

(c) $\displaystyle\lim_{N\to\infty}\frac{|(r_n)_{n=1}^{N}\cap(q\mathbb{N}+\frac{q}{2})|}{N}=\frac{1}{q}$ *(if q is even).*

(ii) For any real-Hilbert space $H_{\mathbb{R}}$, and any unitary operator $U : H_{\mathbb{R}} \to H_{\mathbb{R}}$, and any $f \in H_{\mathbb{R}}$, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle U^{r_n} f, f\rangle = ||Pf||^2, \tag{4.7}$$

where $P : H_{\mathbb{R}} \to H_{\mathbb{R}}$ is the projection onto the largest U-invariant subspace of H.

(iii) $(r_n)_{n=1}^{\infty}$ *is a semi-ergodic sequence.*

Item (iA) of Theorem 4.1.3 naturally leads us to the following definition.

Definition 4.1.4. *A sequence $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$ is* ***uniformly symmetrically distributed*** *if for any finite union of open intervals $U \subseteq [0, 1]$ satisfying $U = 1 - U$ we have*

$$\lim_{N\to\infty} \frac{1}{N}|\{1 \le n \le N \mid x_n \in U\}| = m(U). \tag{4.8}$$

One of our results is an analogue of Weyl's criterion (cf. Theorem 2.4.2) for uniformly symmetrically distributed sequences.

Theorem 4.1.5 (cf. Theorem 4.2.8)**.** *For a sequence $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$ the following are equivalent:*

(i) $(x_n)_{n=1}^{\infty}$ *is uniformly symmetrically distributed.*

(ii) For all $f \in C([0, 1])$ satisfying $f(x) = f(1 - x)$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx. \tag{4.9}$$

(iii) For all $k \in \mathbb{N}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \cos(2\pi i k x_n) = 0. \tag{4.10}$$

We also observe this curious result relating uniform symmetric distribution to van der Corput's difference theorem.

Theorem 4.1.6 (cf. Theorem 4.2.9)**.** *If $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly symmetrically distributed for all $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is uniformly distributed.*

4.2 Main Results

Theorem 4.2.1. *For* $(r_n)_{n=1}^{\infty}$ Z *conditions (i)-(iii) are equivalent.*

(i) For every α R, $(r_n\alpha)_{n=1}^{\infty}$ *is uniformly symmetrically distributed within its orbit closure. More explicitly,*

(A) For every irrational α T, *and every* U T *that is a finite disjoint union of open intervals for which* $U = -U$, *we have*

$$\lim_{N\to\infty}\frac{1}{N}|\{1 \le n \le N \mid r_n\alpha \quad U\}| = m(U). \tag{4.11}$$

(B) For every integer $q \ge 1$, *we have*

(a) $\lim_{N\to\infty}\frac{(r_n)_{n=1}^{N} \cap q\mathsf{N}}{N} = \frac{1}{q}$,

(b) $\lim_{N\to\infty}\frac{(r_n)_{n=1}^{N} \cap ((q\mathsf{N}+j) \quad (q\mathsf{N}-j))}{N} = \frac{2}{q}$, *and*

(c) $\lim_{N\to\infty}\frac{(r_n)_{n=1}^{N} \cap (q\mathsf{N}+\frac{q}{2})}{N} = \frac{1}{q}$ *(if* q *is even).*

(ii) For any real-Hilbert space H_{R}, *and any unitary operator* $U : H_{\mathsf{R}} \to H_{\mathsf{R}}$, *and any* f H_{R}, *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} U^{r_n}f, f \;= ||Pf||^2, \tag{4.12}$$

where convergence takes place in the weak operator topology, and $P : H_{\mathsf{R}} \to H_{\mathsf{R}}$ *is the projection onto the largest* U*-invariant subspace of* H.

(iii) $(r_n)_{n=1}^{\infty}$ *is a semi-ergodic sequence.*

Proof. We will first show that (iii) implies (i). Assuming that (iv) holds, let us first verify that (iB) holds. Let q N be arbitrary, and consider the ergodic m.p.s. $(\{1, 2, \cdots, q\}, \mathcal{B}, \mu, T)$, where $\mathcal{B}$ is the discrete σ-algebra, μ is normalized counting measure, and T is the shift given by $T(x) = x + 1 \pmod q$. Considering $A = \{0\}$, we see that

$$\frac{1}{q^2} = \mu(\{0\})^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(\{0\} \cap T^{r_n}\{0\}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{q}1_{q\mathsf{N}}(r_n), \tag{4.13}$$

which shows that (iBa) holds. Considering $B = \{0, j\}$ for $j = 0, \frac{q}{2}$, we see that

$$\frac{4}{q^2} = \mu(\{0,j\})^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(\{0,j\}\cap T^{r_n}\{0,j\}) \tag{4.14}$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{2}{q}1_{q\mathsf{N}}(r_n) + \frac{1}{q}1_{q\mathsf{N}+j}(r_n) + \frac{1}{q}1_{q\mathsf{N}-j}(r_n)\right),$$

which in conjunction with (iBa), shows us that (iBb) holds. Lastly, if q is even, considering $\{0,\frac{q}{2}\}$ shows us that

$$\frac{4}{q^2} = \mu(\{0,\frac{q}{2}\})^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(\{0,\frac{q}{2}\}\cap T^{r_n}\{0,\frac{q}{2}\}) \tag{4.15}$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{2}{q}1_{q\mathsf{N}}(r_n) + \frac{2}{q}1_{q\mathsf{N}+\frac{q}{2}}(r_n)\right),$$

which in conjunction with (iBa), shows us that (iBc) holds, hence (iB) holds.

To see that (iA) holds, let $\alpha \quad \mathsf{R}\setminus\mathsf{Q}$ be arbitrary. Consider the linear functional ρ : $C([0,\frac{1}{2}])$ that is defined as follows: For $f(x) \quad C([0,\frac{1}{2}])$, let $\tilde{f}(x) \quad C(\mathsf{T})$ be the unique even function that agrees with $f(x)$ on $[0,\frac{1}{2}]$. Consider the ergodic m.p.s. $(\mathsf{T}, \mathcal{B}, m, T)$, where $\mathcal{B}$ is the Lebesgue σ-algebra, m is Lebesgue measure, and T is given by $T(x) = x+\alpha$. For $x \quad (0,\frac{1}{2})$, let $U_x = (0,x)$, $f_x(y) = \mathbb{1}_{[0,x)}(y)(x-|y|)$, and note that $m(U_x \cap T^n U_x) = \tilde{f}_x(n\alpha)$. We now see that for any $x \quad (0,\frac{1}{2})$, we have that

$$\int_{\mathsf{T}}\tilde{f}_x(y)dy = x^2 = m(U_x)^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(U_x\cap T^{r_n}(U_x)) \tag{4.16}$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\tilde{f}_x(r_n\alpha)$$

Since $\mathcal{A} := \{f_x(y)\}_{x \quad (0,\frac{1}{2})} \quad \{\frac{1}{2}\}$ is a collection of continuous functions that separates points on $[0,\frac{1}{2}]$ and vanishes nowhere, the Stone-Weierstrass Theorem tells us that the linear combinations of elements of $\mathcal{A}$ are dense in $C([0,\frac{1}{2}])$, which is identified with the even functions of $C(\mathsf{T})$. It follows that for any $f \quad C([0,\frac{1}{2}])$, we have

$$2\int_0^{\frac{1}{2}} f(y)dm(y) = \int_{\mathsf{T}}\tilde{f}(y)dy = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\tilde{f}_x(r_n\alpha), \tag{4.17}$$

Now let $U \quad (0,\frac{1}{2})$ be an open interval and let $U = U \quad (-U)$. We see that if f_1, f_2 $C([0,1])$ are even functions for which $0 \le f_1 \le \mathbb{1}_U \le f_2$, then

$$\int_0^1 f_1(x)dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_1(r_n\alpha) \leq \lim_{N\to\infty} \frac{1}{N}|\{1 \leq n \leq N \mid r_n\alpha \in U\}| \tag{4.18}$$

$$\leq \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_2(r_n\alpha) = \int_0^1 f_2(x)dx.$$

The desired result follows after using the continuity of the Lebesgue measure to see that

$$m(U) = \sup\{\int_0^1 f_1(x)dx \mid 0 \leq f_1 \leq \mathbb{1}_U\} = \inf\{\int_0^1 f_2(x)dx \mid \mathbb{1}_U \leq f_2\}. \tag{4.19}$$

Let us now show that (i) implies (ii). Since (iA) holds, we see by linearity that for any even step function f, and any irrational $\alpha \in \mathbb{T}$, we have

$$\int_{\mathbb{T}} f(y)dy = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(r_n\alpha). \tag{4.20}$$

Since even Riemann integrable functions can be uniformly approximated by even step functions, we see that the equation above holds for any even Riemann integrable function, such as $\cos(2\pi ikx)$. Let $H_{\mathbb{R}}$ be a real-Hilbert space and let $U : H_{\mathbb{R}} \to H_{\mathbb{R}}$ be a unitary operator. Since $(\langle U^n f, f\rangle)_{n=1}^{\infty}$ is a positive definite sequence we may use Bochner's Theorem to pick a positive measure ν on $\mathbb{T}$ for which $\hat{\nu}(n) = \langle U^n f, f\rangle$ for all $n \in \mathbb{Z}$. Since $\hat{\nu}(n)$ is a real number for all n, we see that

$$\hat{\nu}(n) = \int_{\mathbb{T}} e^{2\pi inx}d\nu = \int_{\mathbb{T}} \cos(2\pi inx)d\nu \tag{4.21}$$

for all $n \in \mathbb{N}$. Furthermore, since $\cos(x)$ is an even Riemann integrable function, and (iA) holds, we see that for all $x \in [0,1] \setminus \mathbb{Q}$, we have that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \cos(2\pi r_n x) = 0. \tag{4.22}$$

Similarly, we see that for $x = \frac{p}{q} \in ([0,1] \cap \mathbb{Q}) \setminus \{0\}$ we may use (iB) to see that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \cos(2\pi r_n \frac{p}{q}) = \sum_{j=1}^{q} \frac{1}{q}\cos(2\pi \frac{pj}{q}) = 0. \tag{4.23}$$

We now see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle U^{r_n}f,f\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{\mathbb{T}}e^{2\pi i r_n x}d\nu \tag{4.24}$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{\mathbb{T}}\cos(2\pi r_n x)d\nu(x)=\int_{\mathbb{T}}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(2\pi r_n x)d\nu(x)$$
$$=\int_{\mathbb{T}}1_{\{0\}}(x)d\nu(x)=\nu(\{0\})=||Pf||^2,$$

which yields the desired result.

Let us now show that (ii) implies (iii). Let $(X,\mathcal{B},\mu,T)$ be an arbitrary ergodic i.m.p.s., let $H_{\mathbb{R}}=L^2(X,\mu)$ (i.e., the real valued elements of $L^2(X,\mu)$), and let $U:H_{\mathbb{R}}\to H_{\mathbb{R}}$ be given by $Uf=f\circ T^{-1}$. We now see that for any $A\in\mathcal{B}$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-r_n}A)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle 1_A,U^{r_n}1_A\rangle=||P1_A||^2=\mu(A)^2. \tag{4.25}$$

□

In order to better compare and contrast semi-ergodic and ergodic sequences in the next section, we require some preliminaries about semi-averaging and averaging sequences.

Definition 4.2.2. *Let $(r_n)_{n=1}^{\infty}\subset\mathbb{Z}$.*

*(i) $(r_n)_{n=1}^{\infty}$ is an **averaging sequence** if for any invertible m.p.s. $(X,\mathcal{B},\mu,T)$ and any $A,B\in\mathcal{B}$ the limit below exists.*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{r_n}B). \tag{4.26}$$

*(ii) $(r_n)_{n=1}^{\infty}$ is an **semi-averaging sequence** if for any invertible m.p.s. $(X,\mathcal{B},\mu,T)$ and any $A\in\mathcal{B}$ the limit below exists.*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{r_n}A). \tag{4.27}$$

Theorem 4.2.3. *For $(r_n)_{n=1}^{\infty}\subset\mathbb{Z}$ conditions (i)-(iii) are equivalent.*

(i) For every $\alpha\in\mathbb{T}$ and every $f\in C(\mathbb{T})$ the limit below exists.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(r_n\alpha). \tag{4.28}$$

(ii) For any Hilbert space $\mathcal{H}$ and any unitary operator $U : \mathcal{H} \to \mathcal{H}$ the limit below exists in the strong operator topology.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} U^{r_n} \tag{4.29}$$

(iii) $(r_n)_{n=1}^{\infty}$ is an averaging sequence.

Proof. Let us first show that (i) implies (ii). Let $f \in \mathcal{H}$ be arbitrary. By the Spectral Theorem (cf. Theorem B.4 in [EW11]) there exists a measure ν on $\mathbb{T}$ and an isomorphism $S : \overline{\text{Span}_{\mathbb{C}}\{U^n f\}_{n=1}^{\infty}} \to L^2(\mathbb{T}, \nu)$ for which $S(U^n f) = e^{2\pi i n x}$. We now see that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} U^n f = S^{-1}\left(\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n x}\right), \tag{4.30}$$

where the existence of the limit on the righthand side in $L^2(\mathbb{T}, \nu)$ follows from the Dominated Convergence Theorem.

To see that (ii) implies (iii), it suffices to consider $\mathcal{H}_{\mathbb{R}} = L^2(X, \mu)$ and $f = \mathbb{1}_A$.

Let us now show that (iii) implies (i). Let $\alpha \in \mathbb{T}$ be arbitrary and consider the m.p.s. $(\mathbb{T}, \mathcal{B}, m, T)$ where m is the Lebesgue measure and $T(x) = x + \alpha$. Let $0 < x, y < \frac{1}{2}$ be arbitrary and observe that for $f_{x,y}(z) = \mathbb{1}_{(y-x,y+x)}(z)(x - |z - y|)$, $A = (0, x)$, and $B = (y, y + x)$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} m(A \cap T^{r_n} B) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_{x,y}(r_n \alpha). \tag{4.31}$$

Since $\mathcal{A} := \{f_{x,y}(z)\}_{x,y \in (0,\frac{1}{2})} \cup \{1\}$ is a collection of continuous functions that separates points on $\mathbb{T}$ and vanishes nowhere, the Stone-Weierstrass Theorem tells us that the linear combinations of elements of $\mathcal{A}$ are dense in $C(\mathbb{T})$. Since

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(r_n \alpha) \tag{4.32}$$

exists for all $f \in \mathcal{A}$, it exists for all $f \in C(\mathbb{T})$ as a result of linearity and standard approximation arguments. □

Theorem 4.2.4. *For $(r_n)_{n=1}^{\infty} \subseteq \mathbb{Z}$ conditions (i)-(iii) are equivalent.*

(i) For every α T *and every* f $C(\mathsf{T})$ *satisfying* $f(x) = f(-x)$ *the limit below exists*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1} N f(r_n \alpha). \tag{4.33}$$

(ii) For any real-Hilbert space H_{R} *and any unitary operator* $U : H \to H$ *the limit below exists.*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \; U^{r_n} f, f \; . \tag{4.34}$$

(iii) $(r_n)_{n=1}^{\infty}$ *is a semi-averaging sequence.*

Since the proof of Theorem 4.2.4 is similar to that of Theorems 4.2.1 and 4.2.3 we omit it. We would now like to investigate sequences analogous to ergodic and semi-ergodic sequences in the more restricted setting of weakly mixing systems. Theorem 4.2.5 provides the desired analogy to ergodic sequences and Theorem 4.2.6 provides the desired an analogy to semi-ergodic sequences. We omit the proof of Theorem 4.2.5 since it is analogous to that of Theorem 4.2.6.

Theorem 4.2.5. *If* $R = (r_n)_{n=1}^{\infty}$ *is a sequence for which* $(r_n\alpha)_{n=1}^{\infty}$ *has an asymptotic distribution function for all but countably many* α T*, then the following are equivalent.*

(i) For all but countably many α T*, the sequence* $(r_n\alpha)_{n=1}^{\infty}$ *is uniformly distributed in its orbit closure.*

(ii) For any Hilbert space H*, and any unitary operator* $U : H \to H$ *with continuous spectrum, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} U^{r_n} = 0, \tag{4.35}$$

with convergence takes place in the strong operator topology.

(iii) For any weakly mixing m.p.s. $(X, \mathcal{B}, \mu, T)$*, and any* A, B $\mathcal{B}$*, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{r_n} B) = \mu(A)\mu(B). \tag{4.36}$$

(iv) For any weakly mixing m.p.s. (X, B, μ, T)*, and any* $A, B \quad \mathsf{B}$*, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\mu(A \cap T^{r_n} B) - \mu(A)\mu(B)| = 0. \tag{4.37}$$

Theorem 4.2.6. *If* $R = (r_n)_{n=1}^{\infty}$ *is a sequence for which* $(r_n \alpha)_{n=1}^{\infty}$ *has an asymptotic distribution function for all but countably many* $\alpha \quad \mathsf{T}$*, then the following are equivalent.*

(i) For all but countably many $\alpha \quad \mathsf{T}$*, the sequence* $(r_n \alpha)_{n=1}^{\infty}$ *is uniformly symmetrically distributed in its orbit closure.*

(ii) For any Hilbert space H*, and any unitary operator* $U : H \to H$ *with continuous spectrum, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \; U^{r_n} f, f \; = 0, \tag{4.38}$$

(iii) For any weakly mixing m.p.s. (X, B, μ, T)*, and any* $A \quad \mathsf{B}$*, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{r_n} A) = \mu(A)^2. \tag{4.39}$$

(iv) For any weakly mixing m.p.s. (X, B, μ, T)*, and any* $A \quad \mathsf{B}$*, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\mu(A \cap T^{r_n} A) - \mu(A)^2| = 0. \tag{4.40}$$

Proof. Let us first show that (i) implies (ii). Let H_{R}, U, P, and f be as in (ii). Since $(\; U^n f, f \;)_{n=1}^{\infty}$ is a positive definite sequence we may use Bochner's Theorem to pick a positive measure ν on T for which $\hat{\nu}(n) = \; U^n f, f \;$ for all $n \quad \mathsf{Z}$. Since $\hat{\nu}(n)$ is a real number for all n, we see that

$$\hat{\nu}(n) = \int_{\mathsf{T}} e^{2\pi i n x} d\nu = \int_{\mathsf{T}} \cos(2\pi i n x) d\nu \tag{4.41}$$

for all $n \quad \mathsf{N}$. Since U has continuous spectrum, we see that ν_f has no atoms. Let $\mathsf{C} \quad \mathsf{T} \setminus \mathsf{Q}$ be the countable collection of α for which $(r_n \alpha)_{n=1}^{\infty}$ is not uniformly symmetrically distributed. Since $\nu(\mathsf{C}) = 0$, we see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \quad f, U^{r_n}f \quad = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\widehat{\nu_f}(n) \tag{4.42}$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{\mathsf{T}}\exp(-2\pi i n x)d\nu_f(x) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{\mathsf{T}}\cos(-2\pi i n x)d\nu_f(x)$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{\mathsf{T}\setminus\mathsf{C}}\cos(-2\pi i n x)d\nu_f(x) = \int_{\mathsf{T}\setminus\mathsf{C}}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(2\pi i n x)d\nu_f(x)$$
$$= \int_{\mathsf{T}\setminus\mathsf{C}}\mathbb{1}_{\{0\}}(x)d\nu_f(x) = 0.$$

Next, let us show that (ii) implies (iii). Let (X, B, μ, T) be an arbitrary weakly mixing m.p.s., and let $H_{\mathsf{R}} = L^2(X,\mu)$. Since T is weakly mixing, we see that $U = U_T$ has continuous spectrum when restricted to $L^2_0(X,\mu) = \{f \quad L^2(X,\mu) \mid \int_X f d\mu = 0\}$. We now see that for $A \quad \mathsf{B}$ we may let $f = \mathbb{1}_A - \mu(A) \quad L^2_0$ to obtain the desired result.

Now let us show that (iii) implies (i). It is only for this implication that we need to use the assumed structure of $(r_n)_{n=1}^{\infty}$. We note that if (X, B, μ, T) is a weakly mixing i.m.p.s., and $A, B \quad \mathsf{B}$ are disjoint, then

$$2\mu(A)\mu(B) = \mu(A \quad B)^2 - \mu(A)^2 - \mu(B)^2 \tag{4.43}$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(\mu((A \quad B)\cap T^{r_n}(A \quad B)) - \mu(A\cap T^{r_n}A) - \mu(B\cap T^{r_n}B))$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\Big(\mu(A\cap T^{r_n}A) + \mu(A\cap T^{r_n}B) + \mu(B\cap T^{r_n}A)$$
$$+ \mu(B\cap T^{r_n}B) - \mu(A\cap T^{r_n}A) - \mu(B\cap T^{r_n}B)\Big)$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(\mu(A\cap T^{r_n}B) + \mu(B\cap T^{r_n}A)).$$

One consequence of this, is that for any $f \quad L^2_{\mathsf{R}}(X,\mu)$, (2) holds. To see this, let us first consider the case in which $f = \sum_{m=1}^{M} c_m \mathbb{1}_{A_m}$ is a simple function. Recalling that U_T^{-1} is also a unitary operator we see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f, U_T^{-r_n} f \;= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{1\le j,k\le M} c_j c_k \; \mathbb{1}_{A_j}, U_T^{-r_n}\mathbb{1}_{A_k} \tag{4.44}$$

$$= \sum_{1\le j<k\le M}\left(\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} c_j c_k(\mu(A_j\cap T^{r_n}A_k)+\mu(A_k\cap T^{r_n}A_j))\right)$$

$$+\sum_{j=1}^{M}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|c_j|^2\mu(A_j\cap T^{r_n}A)$$

$$= \sum_{1\le j<k\le M} 2c_j c_k\mu(A_j)\mu(A_k)+\sum_{j=1}^{M}|c_j|^2\mu(A_j)^2=\int_X |f|^2 d\mu = ||Pf||^2.$$

Now let $\mathcal{C}$ denote the set of $\alpha \quad \mathbb{T}\setminus\mathbb{Q}$ for which $(r_n\alpha)_{n=1}^{\infty}$ is not uniformly symmetrically distributed and let us assume for the sake of contradiction that $\mathcal{C}$ an uncountable set. We see from the definition of $\mathcal{C}$ that

$$\mathcal{C}\setminus\mathbb{Q} = \{\alpha \quad \mathbb{T}\setminus\mathbb{Q} \mid \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(2\pi i r_n\alpha)=0\}. \tag{4.45}$$

Let

$$\mathcal{C}^+ = \{\alpha \quad \mathcal{C}\setminus\mathbb{Q} \mid \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(2\pi i r_n\alpha)>0\},\text{ and } \mathcal{C}^- = \mathcal{C}\setminus\mathcal{C}^+. \tag{4.46}$$

We note that $\mathcal{C}^+$ and $\mathcal{C}^-$ are Borel measurable sets, and that at least one of them is uncountable. Let us first consider the case in which $\mathcal{C}^+$ is uncountable. Since $\mathcal{C}^+$ is an uncountable Borel set, we see that is a measurable isomorphism $S:\mathcal{C}^+\to[0,1]$ (cf. Theorem 1.2.12 in [Par67]), so there exists a continuous Borel probability measure ν that is supported on $\mathcal{C}^+$. By the Gaussian Measure Space Construction, there exists a weakly mixing i.m.p.s. $(X,\mathcal{B},\mu,T)$, and a Gaussian element $f \quad L^2_{\mathbb{R}}(X,\mu)$ for which $f, U_T^n f \;= \widehat{\nu}(n)$ (cf. Chapters 3.11 and 3.12 of [Gla03]). We now see that

$$0=(\int_X f d\mu)^2 = ||Pf||^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f, U_T^{r_n} f \tag{4.47}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{\mathbb{T}}\exp(-2\pi i r_n x)d\nu(x) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{\mathbb{T}}\cos(-2\pi i r_n x)d\nu(x)$$

$$= \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_{\mathbb{T}}\cos(2\pi i r_n x)d\nu(x) \ge \int_{\mathbb{T}}\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(2\pi i r_n x)d\nu(x)$$

$$= \int_{\mathcal{A}^+}\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\cos(2\pi i r_n x)d\nu(x) > 0,$$

which yields the desired contradiction. The case in which C^- is uncountable instead of C^+ is handled similarly.

Lastly, it is clear that (iv) implies (iii), so it only remains to show that (iii) implies (iv). We recall that $(X, \mathcal{B}, \mu, T)$ is weakly mixing if and only if $(X \times X, \mathcal{B} \quad \mathcal{B}, \mu \times \mu, T \times T)$ is weakly mixing. We now see that for any $A \quad \mathcal{B}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{r_n} A) = \mu(A)^2 \text{ and} \tag{4.48}$$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{r_n} A)^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu \times \mu((A \times A) \cap (T^{r_n} \times T^{r_n})(A \times A))$$

$$= \mu(A \times A)^2 = \mu(A)^4, \text{ so}$$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (\mu(A \cap T^{r_n} A) - \mu(A)^2)^2 = 0, \text{ hence}$$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\mu(A \cap T^{r_n} A) - \mu(A)^2| = 0.$$

□

Theorem 4.3.4 shows us that the assumption that $(r_n\alpha)_{n=1}^{\infty}$ have an asymptotic distribution function for all but countably many $\alpha \quad \mathbb{T}$ is a nontrivial assumption, and naturally leads to the following question.

Question 4.2.7. *Does there exist a sequence $(r_n)_{n=1}^{\infty} \quad \mathbb{Z}$ satisfying Theorem 4.2.6 but not Theorem 4.2.5?*

Theorem 4.2.8. *For a sequence $(x_n)_{n=1}^{\infty} \quad [0,1]$ the following are equivalent:*

(i) $(x_n)_{n=1}^{\infty}$ is uniformly symmetrically distributed.

(ii) For all $f \quad C([0,1])$ satisfying $f(x) = f(1-x)$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx. \tag{4.49}$$

(iii) For all $k \quad \mathbb{N}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \cos(2\pi i k x_n) = 0. \tag{4.50}$$

Proof. To see that (i) $\implies$ (ii), we observe that if $U \subseteq [0,1]$ is a finite union of open intervals for which $U = 1 - U$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_U(x_n) = m(U) = \int_0^1 \mathbb{1}_U(x)dx. \tag{4.51}$$

The desired result now follows from the fact that any $f \in C(\mathbb{T})$ satisfying $f(x) = f(-x)$ can be uniformly approximated by linear combinations of step functions $\mathbb{1}_U$ satisfying $\mathbb{1}_U(x) = \mathbb{1}_U(-x)$. It is clear that (ii) $\implies$ (iii), and the fact that (iii) $\implies$ (ii) follows from linearity and the fact that any $f \in C(\mathbb{T})$ can be uniformly approximated by linear combinations of elements from $\{1\} \cup \{\cos(2\pi ikx)\}_{k=1}^{\infty}$. Let us now show that (ii) $\implies$ (i). Let $U \subseteq [0,1]$ be a finite union of open intervals for which $U = 1 - U$. Let $\epsilon > 0$ be arbitrary and let $K \subseteq U$ and $\overline{U} \subseteq W$ be a compact set and an open set respectively for which $m(W \setminus K) < \epsilon$. Let $f_1, f_2 \in C([0,1])$ be such that $f_1(x) = f_1(1-x)$, $f_2(x) = f_2(1-x)$, $f_1(x) = 1$ for $x \in K$, $f_1(x) = 0$ for $x \notin U$, $f_1(x) \in (0,1)$ for $x \in U \setminus K$, $f_2(x) = 1$ for $x \in \overline{U}$, $f_2(x) = 0$ for $x \notin W$, and $f_2(x) \in (0,1)$ for $x \in W \setminus \overline{U}$. We now see that

$$\begin{aligned}\int_0^1 f_1(x)dx = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_1(x_n) \le \liminf_{N\to\infty}\frac{1}{N}|\{1 \le n \le N \mid x_n \in U\}| \\ \le \limsup_{N\to\infty}\frac{1}{N}|\{1 \le n \le N \mid x_n \in U\}| \le \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_2(x_n) = \int_0^1 f_2(x)dx.\end{aligned} \tag{4.52}$$

The desired result now follows from the observation that

$$\left|\int_0^1 f_1(x)dx - m(U)\right| + \left|\int_0^1 f_2(x)dx - m(U)\right| < m(U \setminus K) + m(W \setminus \overline{U}) < \epsilon. \tag{4.53}$$

□

Theorem 4.2.9 is our last result for this section shows how the notion of uniform symmetric distribution can be used to obtain a new variation of van der Corput's difference theorem.

Theorem 4.2.9. *If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly symmetrically distributed for all $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is uniformly distributed.*

Proof. Our proof is a slight modification of the classical proof of van der Corput's Difference Theorem. Let $k \in \mathbb{N}$ and $\epsilon > 0$ both be arbitrary. Let $D \in \mathbb{N}$ be larger than $\frac{1}{\epsilon}$, and note that

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\exp(2\pi ikx_n)\right|^2=\limsup_{N\to\infty}\left|\frac{1}{ND}\sum_{n=1}^{N}\sum_{d=1}^{D}\exp(2\pi ikx_{n+d})\right|^2 \tag{4.54}$$

$$\le\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left|\frac{1}{D}\sum_{d=1}^{D}\exp(2\pi ikx_{n+d})\right|^2$$

$$=\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{D^2}\sum_{d_1,d_2=1}^{D}\exp(2\pi ik(x_{n+d_1}-x_{n+d_2}))$$

$$=\frac{1}{D}+\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{D^2}\sum_{1\le d_1<d_2\le D}(\exp(2\pi ik(x_{n+d_1}-x_{n+d_2}))$$

$$+\exp(2\pi ik(x_{n+d_2}-x_{n+d_1})))$$

$$=\frac{1}{D}+\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{1}{D^2}\sum_{1\le d_1<d_2\le D}2\cos(2\pi k(x_{n+d_2}-x_{n+d_1}))$$

$$\le\frac{1}{D}+\frac{1}{D^2}\sum_{1\le d_1<d_2\le D}\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}2\cos(2\pi k(x_{n+d_2}-x_{n+d_1}))=\frac{1}{D}<\epsilon.$$

□

Theorem 4.2.9 can be interpretted as the statement that in order for $(x_n)_{n=1}^{\infty}$ to be uniformly distributed, it suffices for the set of distances appearing in $(x_{n+h}-x_n)_{n=1}^{\infty}$ to be uniformly distributed for all $h\in\mathbb{N}$. We are naturally led to the following question.

Question 4.2.10. *Does there exist a sequence $(x_n)_{n=1}^{\infty}\subseteq\mathbb{T}$ such that $(x_{n+h}-x_n)_{n=1}^{\infty}$ is uniformly symmetrically distributed for all $h\in\mathbb{N}$ but is not uniformly distributed for some $h\in\mathbb{N}$?*

In light of Chapter 2.4 we are also led to the following question.

Question 4.2.11. *If $(x_n)_{n=1}^{\infty}\subseteq[0,1]$ is such that $(x_{n+h}-x_n)_{n=1}^{\infty}$ is uniformly symmetrically distributed for all $h\in\mathbb{N}$, is $(x_n)_{n=1}^{\infty}$ an o-sequence? Are there analogues of Theorems 2.4.13, 2.4.18, 2.4.20, and 2.4.22 for uniform symmetric distribution?*

4.3 Examples

Theorem 4.3.1.

(i) If $(r_n)_{n=1}^{\infty}\subseteq\mathbb{Z}$ is a semi-ergodic sequence, then for any m.p.s. $(X,\mathcal{B},\mu,T)$ and any $A\in\mathcal{B}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-r_n}A)\geq\mu(A)^2. \tag{4.55}$$

(ii) There exists a sequence $(r_n)_{n=1}^{\infty}\subseteq\mathbb{Z}$ that is not semi-ergodic, but for every m.p.s. $(X,\mathcal{B},\mu,T)$ and every $A\in\mathcal{B}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-r_n}A)\geq\mu(A)^2. \tag{4.56}$$

Proof of (i). Let $\{\mu_e\}_{e\in E}$ denote the set of ergodic T-invariant probability measures on X. By the Ergodic Decomposition, let ν be the probability measure on E, such that for any $A\in\mathcal{B}$, we have

$$\mu(A)=\int_E\mu_e(A)\nu(e). \tag{4.57}$$

We now see that

$$\mu(A)^2=(\int_E\mu_e(A)\nu(e))^2\leq\int_E\mu_e(A)^2\nu(e) \tag{4.58}$$
$$=\int_E\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu_e(A\cap T^{-r_n}))\nu(e)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_E\mu_e(A\cap T^{-r_n}A)$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-r_n}A).$$

□

Proof of (ii). It suffices to take $r_n=2n$. To see that $(r_n)_{n=1}^{\infty}$ is a semi-ergodic sequence, we see that $(n)_{n=1}^{\infty}$ is an ergodic sequence, so for the m.p.s. $(X,\mathcal{B},\mu,T^2)$ we my apply part (i) to see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-r_n}A)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap(T^2)^{-n}A)\geq\mu(A)^2. \tag{4.59}$$

To see that $(r_n)_{n=1}^{\infty}$ is not a semi-ergodic sequence, it suffices to consider the system $(\mathbb{Z}/2\mathbb{Z},\mu,\mathcal{B},T)$, where $\mu(\{0\})=\mu(\{1\})=\frac{1}{2}$ and $T(x)=1-x$. □

Theorem 4.3.2. *There exists a semi-ergodic sequence that is also an averaging sequence, but not an ergodic sequence.*

Proof. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be arbitrary, consider $S = \{n \in \mathbb{N} \mid n\alpha \in [0, \frac{1}{2})\}$, and let $S = (s_n)_{n=1}^{\infty}$. Since $(s_n\alpha)_{n=1}^{\infty}$ is not uniformly distributed in $\mathbb{T}$, we see that $(s_n)_{n=1}^{\infty}$ is not an ergodic sequence, so let us now show that it is a semi-ergodic sequence by verifying that Theorem 4.2.1(i) is satisfied. To do this, we will first check part (iA) of Theorem 4.2.1 by showing that for any irrational $\beta \in \mathbb{T}$ the sequence $(r_n\beta)_{n=1}^{\infty}$ is uniformly symmetrically distributed. If α and β are $\mathbb{Q}$-linearly independent, then the uniform distribution of $(n(\alpha,\beta))_{n=1}^{\infty}$ in $\mathbb{T}^2$ implies that $(s_n\beta)_{n=1}^{\infty}$ is uniformly distributed in $\mathbb{T}$, so it suffices to consider the case $\beta = \frac{p}{q}\alpha = p\frac{1}{q}\alpha$, where $\frac{1}{q}\alpha$ may represent any of the q elements $z \in \mathbb{T}$ satisfying $qz = \alpha$. For each $n \in \mathbb{N}$, let $L_n : \mathbb{T} \to \mathbb{T}$ denote multiplication by n. We see that $s_n\frac{p}{q}\alpha \in U$ if and only if $s_n\frac{1}{q}\alpha \in L_p^{-1}(U)$, but $L_p^{-1}(U)$ is symmetric, and $\mu(U) = \mu(L_p^{-1}(U))$, so it further suffices to consider the case $\beta = \frac{1}{q}\alpha$. We now need the following claim before we can finish the proof of the main Theorem.

Claim. If $U \subseteq \mathbb{T}$, then

$$\{s \in S \mid s\frac{1}{q}\alpha \in U\} = \{n \in \mathbb{N} \mid n\frac{1}{q}\alpha \in U \cap L_q^{-1}([0, \frac{1}{2}))\}. \tag{4.60}$$

Proof of the claim. We see that if $s \in S$ is such that $s\frac{1}{q}\alpha \in U$, then $L_q(s\frac{1}{q}\alpha) = s\alpha \in [0, \frac{1}{2})$, so $s\frac{1}{q}\alpha \in U \cap L_q^{-1}([0, \frac{1}{2}))$, which yields one of the inclusions. To see the reverse inclusion, let $n \in \mathbb{N}$ be such that $n\frac{1}{q}\alpha \in U \cap L_q^{-1}([0, \frac{1}{2}))$, and note that $n\alpha = L_q(n\frac{1}{q}\alpha) \in L_q(U \cap L_q^{-1}([0, \frac{1}{2}))) \subseteq L_q(L_q^{-1}([0, \frac{1}{2}))) = [0, \frac{1}{2})$, so $n \in S$. □

Returning to the proof of Theorem 4.3.2, we see that

$$\begin{aligned} d(\{s \in S \mid s\frac{1}{q}\alpha \in U\}) &= \frac{1}{d(S)}d(\{n \in \mathbb{N} \mid n\frac{1}{q}\alpha \in U \cap L_q^{-1}([0, \frac{1}{2}))\}) \\ =&2d(\{n \in \mathbb{N} \mid n\frac{1}{q}\alpha \in U \cap L_q^{-1}([0, \frac{1}{2}))\}) = 2\mu(U \cap L_q^{-1}([0, \frac{1}{2}))) \\ =&\mu(U \cap L_q^{-1}([0, \frac{1}{2}))) + \mu(-(U \cap L_q^{-1}([0, \frac{1}{2})))) \\ =&\mu(U \cap L_q^{-1}([0, \frac{1}{2}))) + \mu(-U \cap -L_q^{-1}([0, \frac{1}{2}))) \\ =&\mu(U \cap L_q^{-1}([0, \frac{1}{2}))) + \mu(U \cap L_q^{-1}(-[0, \frac{1}{2}))) \\ =&\mu(U \cap L_q^{-1}([0, \frac{1}{2}))) + \mu(U \cap L_q^{-1}([\frac{1}{2}, 1))) = \mu(U \cap L_q^{-1}([0, 1))) \\ =&\mu(U \cap [0, 1)) = \mu(U), \end{aligned} \tag{4.61}$$

so Theorem 4.2.1(iA) is satisfied.

Next, we will check part (iB) of Theorem 4.2.1 by showing that if $\frac{p}{q} \in \mathbb{Q} \cap \mathbb{T}$, then $(s_n \frac{p}{q})_{n=1}^{\infty}$ is uniformly distributed in its orbit. It suffices to show that $(s_n)_{n=1}^{\infty}$ is uniformly distributed modulo q. To see that this is the case, we note that for any $r \in [0, q)$, we have $(qn + r)\alpha \in [0, \frac{1}{2})$ if and only if $n\alpha \in L_q^{-1}([0, \frac{1}{2}) - r\alpha)$. Since $\mu(L_q^{-1}([0, \frac{1}{2}) - r\alpha)) = \mu([0, \frac{1}{2})) = \frac{1}{2}$, we see that $d(\{n \in \mathbb{N} | (qn+r)\alpha \in [0, \frac{1}{2})\}) = \frac{1}{2}$, so $d(n \in \mathbb{N} | n\alpha \in [0, \frac{1}{2}) \& \ n \equiv r \pmod q\}) = \frac{1}{2q}$, which shows us that $d(n \in S | \ n \equiv r \pmod q\}) = \frac{1}{q}$.

Now that we have shown $(s_n)_{n=1}^{\infty}$ is a semi-ergodic but not ergodic sequence, it only remains to show that it is an averaging sequence. We note that the claim tells us that for any $0 \leq x \leq 1$ and $\frac{p}{q} \in \mathbb{Q} \setminus \{0\}$, the limit below exists.

$$\lim_{N \to \infty} \frac{1}{N} |\{1 \leq n \leq N \mid s_n \frac{p}{q}\alpha \in [0, x)\}|. \tag{4.62}$$

Since $(s_n \beta)_{n=1}^{\infty}$ is uniformly distributed within its orbit closure for all β that are rationally independent from α, we see that Theorem 4.2.3(i) is satisfied, so $(s_n)_{n=1}^{\infty}$ is an averaging sequence. □

Remark 4.3.3. We see that the construction above can be generalized as follows to provide a family of examples of semi-ergodic sequences that are also averaging sequences, but not ergodic sequences. Let $\{U_j\}_{j=1}^{N}$ be a collection of open intervals in $\mathbb{T}$, such that $m(\bigcup_{j=1}^{N}(U_j \cup -U_j)) = 1$, and $\{U_j\}_{j=1}^{N} \cup \{-U_j\}_{j=1}^{N}$ is a collection of mutually disjoint sets. Then for any irrational $\alpha \in \mathbb{T}$, the sequence $\{n \in \mathbb{N} \mid \exists j$ s.t. $n\alpha \in U_j\}$ is semi-ergodic and averaging, but not ergodic.

Theorem 4.3.4. *There exists a semi-ergodic sequence $(r_n)_{n=1}^{\infty} \subseteq \mathbb{Z}$ that is not an averaging sequence. Moreover, the set of $\alpha \in \mathbb{T}$ for which*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \sin(2\pi i r_n \alpha) \tag{4.63}$$

does not exist is uncountable and dense in $\mathbb{T}$.

Proof. Our proof is motivated by the proof of Theorem 11 in [ET57]. Let $(\alpha_n)_{n=1}^{\infty} \subseteq \mathbb{T} \setminus \mathbb{Q}$ be dense in $\mathbb{T}$. Let $(\beta_n)_{n=1}^{\infty}$ be an enumeration of $(\alpha_n)_{n=1}^{\infty}$ in which each α_n occurs infinitely many times. We will inductively define a sequence of positive integers $(t_k)_{k=1}^{\infty}$, and subsequently define r_n for $n \in [t_k, t_{k+1})$. For the base case, let $t_1 = 1$. For the inductive step, we recall that $(n\beta_k)_{n=t_k}^{\infty}$ is a uniformly distributed sequence, hence

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=t_k}^{t_k+N} |\sin(2\pi n \beta_k)| = \int_0^1 |\sin(2\pi x)| dx = \frac{2}{\pi}, \tag{4.64}$$

so let t_{k+1} be such that $t_{k+1} \geq 5t_k$, and

$$\frac{1}{t_{k+1} - t_k} \sum_{n=t_k}^{t_{k+1}-1} |\sin(2\pi n \beta_k)| \geq \frac{1}{\pi}. \tag{4.65}$$

Let $s \quad \mathbb{N}$ be such that $\beta_k = \alpha_s$, and suppose that this is the wth time that α_s has occured so far. For $n \quad [t_k, t_{k+1})$, let

$$r_n = \begin{cases} (-1)^{w+1} n & \text{if } \sin(2\pi n \beta_k) \geq 0 \\ (-1)^w n & \text{else} \end{cases}. \tag{4.66}$$

We see that the sequence $(r_n)_{n=1}^{\infty}$ is a semi-ergodic sequence since it satisfies Theorem 4.2.1(i). To see that $(r_n)_{n=1}^{\infty}$ is not an averaging sequence we will show that Theorem 4.2.3(i) is not satisfied. We note that for each $k \quad \mathbb{N}$, we have

$$|\frac{1}{t_{k+1} - 1} \sum_{n=1}^{t_{k+1}-1} \sin(2\pi r_n \beta_k)| \geq |\frac{1}{t_{k+1} - 1} \sum_{n=t_k}^{t_{k+1}-1} \sin(2\pi r_n \beta_k)| - \frac{t_k - 1}{t_{k+1} - 1} \tag{4.67}$$
$$= \frac{1}{t_{k+1} - 1} \sum_{n=t_k}^{t_{k+1}-1} |\sin(2\pi r_n \beta_k)| - \frac{t_k - 1}{t_{k+1} - 1} \geq \frac{t_{k+1} - t_k}{\pi(t_{k+1} - 1)} - \frac{t_k - 1}{t_{k+1} - 1}$$
$$= \frac{t_{k+1} - t_k - \pi(t_k - 1)}{\pi(t_{k+1} - 1)} \geq \frac{t_{k+1} - t_k - \pi(t_k - 1)}{\pi t_{k+1}} \geq \frac{4 - \pi}{\pi} > 0.$$

In particular, if the value of w corresponding to k is odd, then

$$\frac{1}{t_{k+1} - 1} \sum_{n=1}^{t_{k+1}-1} \sin(2\pi r_n \beta_k) \geq \frac{4 - \pi}{\pi}, \tag{4.68}$$

and if the value of w corresponding to k is even, then

$$\frac{1}{t_{k+1} - 1} \sum_{n=1}^{t_{k+1}-1} \sin(2\pi r_n \beta_k) \leq -\frac{4 - \pi}{\pi}. \tag{4.69}$$

By the continuity of sin, for each $k \quad \mathbb{N}$, let I_k be an open interval centered at β_k, such that for any $\gamma \quad I_k$, we have

$$|\frac{1}{t_{k+1} - 1} \sum_{n=1}^{t_{k+1}-1} \sin(2\pi r_n \gamma) - \frac{1}{t_{k+1} - 1} \sum_{n=1}^{t_{k+1}-1} \sin(2\pi r_n \alpha_k)| \leq \frac{4 - \pi}{2\pi}. \tag{4.70}$$

Let $E = \cap_{m=1}^{\infty} \quad {}_{k=m}^{\infty} I_k$, and note that $(\alpha_n)_{n=1}^{\infty} \quad E$, so E is a dense G_δ, which must be uncountable. Moreover, for each $\gamma \quad E$, we have that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \sin(2\pi r_n \gamma) \geq \frac{4 - \pi}{2\pi}, \text{ and} \tag{4.71}$$

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \sin(2\pi r_n \gamma) \leq -\frac{4 - \pi}{2\pi}.$$

□

CHAPTER 5

VAN DER CORPUT SETS

5.1 Introduction

We will assume that the reader of this section is familiar with van der Corput's Difference Theorem (cf. Theorem 2.1.2), and for our results involving nearly mixing sequences we will assume that the reader is familiar with Chapter 2.2. A natural way in which we can improve van der Corput's Difference Theorem is through the notion of van der Corput sets.

Definition 5.1.1. *$R \subseteq \mathbb{N}$ is a **van der Corput set (vdC set)** if for any $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ for which $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for all $h \in R$ we have that $(x_n)_{n=1}^{\infty}$ is uniformly distributed.*

Interestingly, vdC sets have many equivalent characterizations.

Theorem 5.1.2. *For $R \subseteq \mathbb{N}$ the following are equivalent.*

(i) R is a vdC set.

(ii) If $(x_n)_{n=1}^{\infty}$ is a sequence of complex numbers of norm 1 satisfying

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_{n+h}\overline{x_n} = 0 \tag{5.1}$$

for all $h \in R$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n = 0. \tag{5.2}$$

(iii) For any Hilbert space H, any unitary operator $U : H \to H$, and any $f \in H$ satisfying $\langle U^r f, f\rangle = 0$ for every $r \in R$, then

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} U^d f|| = 0. \tag{5.3}$$

(iv) *For any m.p.s* $(X, \mathcal{B}, \mu, T)$ *and any* $f \in L^2(X, \mu)$ *satisfying* $\langle U_T^r f, h\rangle = 0$ *for every* $h \in H$*, then* $\int_X f d\mu = 0$*.*

(v) *If* μ *is a probability measure on* $\mathbb{T}$ *for which* $\hat{\mu}(r) = 0$ *for all* $r \in R$*, then* $\mu(\{0\}) = 0$*.*

(vi) *If* μ *is a probability measure on* $\mathbb{T}$ *for which* $\hat{\mu}(r) = 0$ *for all* $r \in R$*, then* μ *is continuous.*

(vii) *For each* $\epsilon > 0$ *there exists a trigonometric polynomial* $P_\epsilon : \mathbb{T} \to [-\epsilon, \infty)$ *such that* $\hat{P}_\epsilon(n) = 0$ *if* $n \in \mathbb{Z} \setminus (R \cup -R)$ *and* $P(0) = 1$*.*

(viii) *If* $\mathcal{H}$ *is a Hilbert space and* $(x_n)_{n=1}^\infty \subseteq SA(\mathcal{H})$ *satisfies*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0, \tag{5.4}$$

for every $h \in R$*, then*

$$\lim_{N \to \infty} \left|\left|\frac{1}{N} \sum_{n=1}^{N} x_n\right|\right| = 0. \tag{5.5}$$

Proof. The fact that (vii) $\implies$ (i) was first shown in [KMF78]. The equivalence of (i), (ii), (v), and (vii) can be deduced from [Ruz84], and another exposition of these equivalences is given in [BL08]. It is clear that (viii) $\implies$ (ii), and the fact that (i) $\implies$ (viii) can be deduced from Theorem 5.1.9. It is clear that (vi) $\implies$ (v), and to see that (v) $\implies$ (vi) it suffices to observe that if μ_t is the measure given by $\mu_t(E) = \mu(E + t)$, then $\mu_t(\{0\}) = \mu(\{t\})$ and $\hat{\mu}_t(n) = e^{2\pi i n t}\hat{\mu}(n)$. The equivalence of (i) and (iii) can be deduced from [NRS12] after an application of the mean ergodic theorem. It is clear that (iii) $\implies$ (iv), and the fact that (iv) $\implies$ (i) can be deduced from the fact that (iv) implies item (iv) of Theorem 5.1.3. □

The first main result of this chapter is to add to the list of characterizations of vdC sets. We remark that our additional characterizations relate to those of Theorem 5.1.2 in the same way that uniform symmetric distribution relates to uniform distribution, and that a few of them were already observed in [Ruz84]. For a measure μ on $\mathbb{T}$ we define the measure $\tilde{\mu}$ by $\tilde{\mu}(E) = \mu(-E)$ for all measurable $E \subseteq \mathbb{T}$.

Theorem 5.1.3 (cf. Theorem 5.2.2)**.** *For* $R \subseteq \mathbb{N}$ *the following are equivalent.*

(i) R *is a vdC set.*

(ii) *If* $(x_n)_{n=1}^\infty \subseteq [0, 1]$ *is such that* $(x_{n+h} - x_n)_{n=1}^\infty$ *is uniformly symmetrically distributed for all* $h \in R$*, then* $(x_n)_{n=1}^\infty$ *is uniformly symmetrically distributed.*

(iii) If $(x_n)_{n=1}^{\infty}$ $[-1,1]$ *is such that*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_{n+h}x_n = 0 \tag{5.6}$$

for all h R*, then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n = 0. \tag{5.7}$$

(iv) If μ *is a probability measure on* T *for which* $\mu = \tilde{\mu}$ *and* $\hat{\mu}(h) = 0$ *for all* h R*, then* $\mu(\{0\}) = 0$*.*

(v) If μ *is a probability measure on* T *for which* $\mu = \tilde{\mu}$ *and* $\hat{\mu}(h) = 0$ *for all* h R*, then* μ *is continuous.*

(vi) For each $\epsilon > 0$ *there exists a trigonometric polynomial* $P_\epsilon : \mathsf{T} \to [-\epsilon, \infty)$ *of the form* $P_\epsilon(x) = \sum_{n=1}^{N} a_n \cos(2\pi i n x)$ *such that* $\hat{P}_\epsilon(n) = 0$ *if* n $\mathsf{Z} \setminus (D \quad -D)$ *and* $P(0) = 1$*.*

(vii) For any sequence of unit vectors $(x_n)_{n=1}^{\infty}$ R^2 *satisfying*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \quad x_{n+h}, x_n \quad = 0, \tag{5.8}$$

for every h R*, we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n = 0. \tag{5.9}$$

Another interesting feature of vdC sets is their relationship with sets of recurrence.

Definition 5.1.4. *Let* R N*.*

(i) R *is a set of* ***measurable recurrence*** *if for every m.p.s.* $(X, \mathcal{B}, \mu, T)$ *and every* A $\mathcal{B}$ *with* $\mu(A) > 0$ *there exists* r R *for which*

$$\mu(A \cap T^{-1}A) > 0. \tag{5.10}$$

(ii) *R is a set of **strong recurrence** if for every m.p.s. $(X, \mathcal{B}, \mu, T)$ and every $A \in \mathcal{B}$ with $\mu(A) > 0$ we have*

$$\limsup_{\substack{r\to\infty \\ r\in R}} \mu(A \cap T^{-r}A) > 0. \tag{5.11}$$

(iii) *$R = (r_n)_{n=1}^{\infty}$ is a set of **averaging recurrence**[1] if for every m.p.s. $(X, \mathcal{B}, \mu, T)$ and every $A \in \mathcal{B}$ with $\mu(A) > 0$ we have*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-r_n}A) > 0. \tag{5.12}$$

(iv) *R is a set of **nice recurrence** if for every m.p.s. $(X, \mathcal{B}, \mu, T)$, every $A \in \mathcal{B}$ with $\mu(A) > 0$, and every $\epsilon > 0$ there exist infinitely many $r \in R$ for which*

$$\mu(A \cap T^{-r}A) > \mu(A)^2 - \epsilon. \tag{5.13}$$

(v) *R is a set of **operator recurrence** if for any Hilbert space $\mathcal{H}$, any unitary operator $U : \mathcal{H} \to \mathcal{H}$, and any $x \in \mathcal{H}$ with $Px \neq 0$ (where $P : \mathcal{H} \to \mathcal{H}$ is the orthogonal projection onto $\ker(U - I)$), there is some $r \in R$ for which $\langle U^r x, x\rangle \neq 0$.*[2]

In [KMF78] it is shown that any vdC set is a set of measurable recurrence, and in [NRS12] it is shown that vdC sets are in fact equivalent to sets of operator recurrence (Use the Mean Ergodic Theorem and Theorem 5.1.2(iii)). In [Ruz82] Ruzsa asked whether or not every set of measurable recurrence is also a vdC set and Bourgain answered this question in the negative in [Bou87]. In order to continue this line of results we need a few more definitions of generalizations of vdC sets using the form of Theorem 5.1.2(ii) when $\mathcal{H} = \mathbb{C}$.

Definition 5.1.5. *Let $R \subseteq \mathbb{N}$.*

(i) *R is an **enhanced van der Corput (vdC) set** if for any sequence of complex numbers $(y_n)_{n=1}^{\infty}$ of modulus 1 satisfying*

$$\lim_{\substack{r\to\infty \\ r\in R}} \limsup_{N\to\infty} \left| \frac{1}{N}\sum_{n=1}^{N} y_{n+r}\overline{y_n} \right| = 0, \tag{5.14}$$

[1] We are using the definition from [BL08] which differs from the definition in [BH96] since we are using a lim sup rather than a lim.

[2] Our definition differs from that in [NRS12] since we use U^r rather than U^{-r}, but the definitions are equivalent since U^{-1} is also a unitary operator.

we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} y_n = 0. \tag{5.15}$$

(ii) *R is a **density van der Corput (vdC) set** if for any sequence of complex numbers $(y_n)_{n=1}^{\infty}$ of modulus 1 satisfying*

$$\lim_{M\to\infty} \frac{1}{M}\sum_{n=1}^{M} \limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} y_{n+r_m}\overline{y_n}\right| = 0, \tag{5.16}$$

we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} y_n = 0. \tag{5.17}$$

(iii) *R is a **nice van der Corput (vdC) set** if for any sequence of complex numbers $(y_n)_{n=1}^{\infty}$ of modulus 1 we have*

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} y_n\right|^2 \le \limsup_{\substack{r\to\infty\\ r\in R}} \limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} y_{n+r}\overline{y_n}\right|. \tag{5.18}$$

In [BL08] it is shown that every enhanced vdC set is a set of strong recurrence and that every density vdC set is a set of averaging recurrence. It is also asked whether or not every nice vdC set is a set of nice recurrence, and one of the main results of this chapter is to answer this question in the positive.

Theorem 5.1.6 (cf. Theorem 5.2.4)**.** *Every nice vdC set is a set of nice recurrence.*

While the methods of Theorem 2 of [KMF78] can be modified to attain this result, we will provide a slightly more constructive proof. We remark that it is still open as to whether or not there exists a set of strong recurrence or a set of nice recurrence that is not a vdC set. Before discussing our next result we require some more definitions motivated by item (vi) of Theorem 5.1.2.

Definition 5.1.7. *Let $R \subseteq \mathbb{N}$.*

(i) R *is a* $\boldsymbol{FC^+}$ *set if every positive measure* μ *on* $\mathbb{T}$ *satisfying*

$$\lim_{\substack{r \to \infty \\ r \in R}} \hat{\mu}(r) = 0 \tag{5.19}$$

is continuous.

(ii) $R = (r_m)_{m=1}^{\infty}$ *is a* ***density*** $\boldsymbol{FC^+}$ *set if every positive measure* μ *on* $\mathbb{T}$ *satisfying*

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} |\hat{\mu}(r_m)| = 0 \tag{5.20}$$

is continuous.[3]

(iii) R *is a* ***nice*** $\boldsymbol{FC^+}$ *set if every positive measure* μ *on* $\mathbb{T}$ *we have*

$$\sup_{r \in R} |\hat{\mu}(r)| \geq \mu(\{0\}). \tag{5.21}$$

In [BL08] it is shown that every nice FC$^+$ set is a nice vdC set and it is asked whether or not the converse is true. We answer this question in the positive as part of our next result.

Theorem 5.1.8 (cf. Theorem 5.2.5)**.** *For* $R \subseteq \mathbb{N}$ *the following are equivalent.*

(i) R *is a nice FC*$^+$ *set.*

(ii) R *is a nice vdC set.*

(iii) *For every Hilbert space* $\mathcal{H}$*, every unitary operator* $U : \mathcal{H} \to \mathcal{H}$*, and every* $f \in \mathcal{H}$*, we have*

$$\sup_{r \in R} |\langle U^r f, f \rangle| \geq ||Pf||^2, \tag{5.22}$$

where P *is the projection onto the subspace of* U*-invariant elements.*

[3] We observe that our definition differs from Definition 7 of [BL08] since we use $|\hat{\mu}(r_m)|$ instead of $\hat{\mu}(r_m)$. This is because $\mathbb{N}$ does not satisfy Definition 7 of [BL08] as seen by letting μ be a point mass at any point of $\mathbb{T}$ other than 0. It is for this reason that we believe Definition 7 of [BL08] has a typo.

(iv) *For any positive measure μ on $\mathbb{T}$ we have*

$$\sup_{r \in R} |\hat{\mu}(r)|^2 \geq \sum_{t \in A} \mu(\{t\})^2, \tag{5.23}$$

where $A \subset \mathbb{T}$ is the set of atoms of μ.

(v) *For every Hilbert space $\mathcal{H}$, every unitary operator $U : \mathcal{H} \to \mathcal{H}$, and every $f \in \mathcal{H}$, we have*

$$\sup_{r \in R} |\langle U^r f, f \rangle| \geq ||P_c f||^2, \tag{5.24}$$

where $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{wm}$ is the Jacobs-de Leeuw-Glicksberg decomposition (cf. Theorem 2.3.5) and P_c is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_c$.

We remark that Theorem 5.1.8 has a symmetric analogue similar to Theorem 5.1.3, but we do not state it here for the sake of brevity. The next set of main results of this chapter connect vdC sets to the notion of weak mixing from the ergodic hierarchy of mixing. They are similar in spirit to items (iv) and (v) of Theorem 5.1.8. We remind the reader that we will be assuming familiarity with the notion of nearly mixing sequences in a Hilbert space $\mathcal{H}$ from Definition 2.2.5 when discussing Theorem 5.1.9. Theorem 5.1.10 is equivalent to Theorem 5.1.9 and does not require any knowledge about nearly mixing sequences.

Theorem 5.1.9 (cf. Theorem 5.2.6)**.** *Let $R \subset \mathbb{N}$, $\mathcal{H}$ be a Hilbert space, and $(y_n)_{n=1}^{\infty} \subset SA(\mathcal{H})$.*

(i) *If R is a vdC set and*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle y_{n+r}, y_n \rangle = 0 \tag{5.25}$$

for all $r \in R$, then $(y_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(ii) *If R is an enhanced vdC set and*

$$\lim_{\substack{r \to \infty \\ r \in R}} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle y_{n+r}, y_n \rangle \right| = 0, \tag{5.26}$$

then $(y_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(iii) If $R = (r_n)_{n=1}^{\infty}$ *is a density vdC set and*

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle y_{n+r_m}, y_n\rangle\right| = 0, \tag{5.27}$$

then $(y_n)_{n=1}^{\infty}$ *is a nearly weakly mixing sequence.*

(iv) In (ii) and (iii) is is possible for $(y_n)_{n=1}^{\infty}$ *to be a rigid sequence. In particular,* $(y_n)_{n=1}^{\infty}$ *need not be a nearly mildly mixing sequence.*

Theorem 5.1.10 (cf. Theorem 5.2.9)**.** *Let* $R \subseteq \mathbb{N}$*,* $\mathcal{H}$ *be a Hilbert space,* $U : \mathcal{H} \to \mathcal{H}$ *a unitary operator, and let* $x \in \mathcal{H}$*. Let* $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{wm}$ *be the Jacobs-de Leeuw-Glicksberg decomposition and let* P_c *be the orthogonal projection of* $\mathcal{H}$ *onto* $\mathcal{H}_c$*.*

(i) If R *is a vdC set and* $\langle U^r x, x\rangle = 0$ *for all* $x \in R$*, then* $P_c x = 0$*.*

(ii) If R *is an enhanced vdC set and*

$$\lim_{\substack{r\to\infty \\ r\in R}} \langle U^r x, x\rangle = 0, \tag{5.28}$$

then $P_c x = 0$*.*

(iii) If $R = (r_n)_{n=1}^{\infty}$ *is a density vdC set and*

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\langle U^{r_m} x, x\rangle = 0, \tag{5.29}$$

then $P_c x = 0$*.*

(iv) In (ii) and (iii) it is possible for x *to be a rigid element of* $(\mathcal{H}, U)$*. In particular,* x *need not be a mildly mixing element of* $(\mathcal{H}, U)$*.*

Our last result is similar to the previous 2 results in the context of uniform distribution. We remind the reader that we will be assuming familiarity with the notion of wm-sequences from Definition 2.4.6 when discussing Theorem 5.1.11.

Theorem 5.1.11 (cf. Theorem 5.2.10)**.** *Let* $R \subseteq \mathbb{N}$*,* $(x_n)_{n=1}^{\infty} \subseteq [0,1]$*, and let* $\overline{D}$ *be the measure of discrepancy discussed in Definition 2.4.3.*

(i) If R *is a vdC set and* $(x_{n+r} - x_n)_{n=1}^{\infty}$ *is uniformly distributed for all* $r \in R$*, then* $(x_n)_{n=1}^{\infty}$ *is a wm-sequence.*

(ii) If R is an enhanced vdC set and

$$\lim_{\substack{r\to\infty \\ r\in R}} \overline{D}((x_{n+r}-x_n)_{n=1}^{\infty}) = 0, \tag{5.30}$$

then $(x_n)_{n=1}^{\infty}$ is a wm-sequence.

(iii) If $R=(r_n)_{n=1}^{\infty}$ is a density vdC set and

$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \overline{D}((x_{n+r_m}-x_n)_{n=1}^{\infty}) = 0, \tag{5.31}$$

then $(x_n)_{n=1}^{\infty}$ is a wm-sequence.

(iv) In (ii) and (iii) $(x_n)_{n=1}^{\infty}$ need not be an mm-sequence. In fact, it is possible to have $(f(x_n))_{n=1}^{\infty}$ be a rigid sequence for all $f \in C([0,1])$ satisfying $\int_0^1 f(x)dx = 0$.

5.2 Main Results

We begin with a useful lemma about positive definite sequences.

Lemma 5.2.1. *Let μ be a positive measure on $\mathbb{T}$.*

(i) If μ is discrete then there exists an ergodic m.p.s. $(X,\mathcal{B},\mu,T)$ that is isomorphic to a rotation on a compact abelian group and a $f \in L^2(X,\mu)$ for which $\hat{\mu}(n) = \langle U^n f, f\rangle$ for all $n \in \mathbb{N}$.

(ii) If μ is continuous then there exists a weakly mixing m.p.s. $(X,\mathcal{B},\mu,T)$ and $f \in L^2(X,\mu)$ for which $\hat{\mu}(n) = \langle U^n f, f\rangle$ for all $n \in \mathbb{N}$.

(iii) There exists an ergodic m.p.s. $(X,\mathcal{B},\mu,T)$ and a $f \in L^2(X,\mu)$ for which $\hat{\mu}(n) = \langle U^n f, f\rangle$ for all $n \in \mathbb{N}$.

Proof of (i). Let $A \subseteq \mathbb{T}$ denote the support of μ_d. Consider $g = (a_j)_{a_j\in A} \in \mathbb{T}^A$, let X denote the closure of $\{g^n\}_{n\in\mathbb{Z}}$, let $\mathcal{B}$ denote the Borel σ-algebra restricted to X, and let ν denote the normalized Haar measure on X. Let $T: X\to X$ be given by $Tx = gx$, and note that (X,g) is a minimal group rotation, so $(X,\mathcal{B},\nu,T)$ is an ergodic m.p.s. For $a_j \in A$ we see that $f_j(x_1,\cdots,x_j,\cdots) = e^{2\pi i x_j} \in L^2(X,\nu)$ is an eigen function for the eigen value $e^{2\pi i a_j}$. We now see that for $f := \sum_{a_j\in A}\sqrt{\mu(\{a_j\})}f_j$ and any $n \in \mathbb{N}$ we have

$$\langle U^n f, f\rangle = \int_X \sum_{a_j\in A} \mu(\{a_j\}) e^{2\pi i(x_j+na_j)} e^{-2\pi i x_j} d\nu(x) = \sum_{a_j\in A} e^{2\pi i n a_j}\mu(\{a_j\}) = \hat{\mu}(n) \tag{5.32}$$

□

Proof of (ii). This is a consequence of the Gaussian measure space construction as discussed in Chapters 3.11 and 3.12 of [Gla03]. We refer the reader to Chapter 1.4 of [Jan97] to see why the considerations of [Gla03] for real-valued Gaussian processes are also valid for complex-valued Gaussian processes. □

Proof of (iii). We remark that (iii) is not an immediate consequence of the Gaussian measure space construction since a Gaussian measure space is ergodic if and only if it is weakly mixing, both of which occur if and only if the spectral measure is continuous. Let $\mu = \mu_d + \mu_c$ where μ_d is a discrete measure and μ_c is a continuous measure. Using parts (i) and (ii), for $k \in \{d, c\}$ let $(X_k, \mathcal{B}_k, \nu_k, T_k)$ be an ergodic m.p.s. and let $f_k \in L^2(X_k, \nu_k)$ be such that $\hat{\mu_k}(n) = \langle U_k^n f_k, f_k \rangle$ for all $n \in \mathbb{N}$. Since $(X_c, \mathcal{B}_c, \nu_c, T_c)$ is aso weakly mixing, we see that $(X_d \times X_c, \mathcal{B}_d \otimes \mathcal{B}_c, \nu_d \times \nu_c, T_d \times T_c)$ is an ergodic m.p.s. Consider $\tilde{f}_d(x, y) = f_d(x)$ and $\tilde{f}_c(x, y) = f_c(y)$ and observe that $f := \tilde{f}_d + \tilde{f}_c \in L^2(X_d \times X_c, \nu_d \times \nu_c)$. Since μ_c is continuous, we see that $\int_{X_c} f_c d\mu_c = 0$, hence

$$\begin{aligned} \langle (U_d \otimes U_c)^n \tilde{f}_c, \tilde{f}_d \rangle &= \left(\int_{X_c} U_c^n f_c d\mu_c\right)\left(\int_{X_d} f_d d\mu_d\right) = 0, \text{ and} \\ \langle (U_d \otimes U_c)^n \tilde{f}_d, \tilde{f}_c \rangle &= \left(\int_{X_d} U_d^n f_d d\mu_d\right)\left(\int_{X_c} f_c d\mu_c\right) = 0. \end{aligned} \tag{5.33}$$

We now see that for all $n \in \mathbb{N}$ we have

$$\begin{aligned} \langle (U_d \otimes U_c)^n f, f \rangle &= \langle U_d^n f_d, f_d \rangle + \langle (U_d \otimes U_c)^n \tilde{f}_d, \tilde{f}_c \rangle \\ &\quad + \langle (U_d \otimes U_c)^n \tilde{f}_c, \tilde{f}_d \rangle + \langle U_c^d f_c, f_c \rangle \\ &= \hat{\mu_d}(n) + \hat{\mu_c}(n) = \hat{\mu}(n). \end{aligned} \tag{5.34}$$

□

Theorem 5.2.2. *For $R \subseteq \mathbb{N}$ the following are equivalent.*

(i) R *is a vdC set.*

(ii) If $(x_n)_{n=1}^{\infty} \subseteq [-1, 1]$ *is such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_{n+h} x_n = 0 \tag{5.35}$$

for all $h \in R$*, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 0. \tag{5.36}$$

(iii) *If* $(x_n)_{n=1}^{\infty}$ $[0,1]$ *is such that* $(x_{n+h} - x_n)_{n=1}^{\infty}$ *is uniformly symmetrically distributed for all* h R*, then* $(x_n)_{n=1}^{\infty}$ *is uniformly symmetrically distributed.*

(iv) *For any m.p.s* (X, B, μ, T) *and any real-valued* f $L^2(X,\mu)$ *satisfying* $U_T^h f, f = 0$ *for every* h H*, then* $\int_X f d\mu = 0$*.*

(v) *If* μ *is a probability measure on* T *for which* $\mu = \tilde{\mu}$ *and* $\hat{\mu}(h) = 0$ *for all* h R*, then* $\mu(\{0\}) = 0$*.*

(vi) *If* μ *is a probability measure on* T *for which* $\mu = \tilde{\mu}$ *and* $\hat{\mu}(h) = 0$ *for all* h R*, then* μ *is continuous.*

(vii) *For each* $\epsilon > 0$ *there exists a trigonometric polynomial* $P_\epsilon : T \to [-\epsilon, \infty)$ *of the form* $P_\epsilon(x) = \sum_{n=1}^{N} a_n \cos(2\pi inx)$ *such that* $\hat{P}_\epsilon(n) = 0$ *if* n $Z \setminus (D \quad -D)$ *and* $P(0) = 1$*.*

(viii) *For any sequence of unit vectors* $(x_n)_{n=1}^{\infty}$ R^2 *satisfying*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_{n+h}, x_n = 0, \tag{5.37}$$

for every h R*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 0. \tag{5.38}$$

Proof. We begin by observing that $\mu = \tilde{\mu}$ if and only if $\hat{\mu}(n) = \int_T \cos(2piinx)dx$ for all n N. In light of this observation, the equivalence of (i), (v), and (vii) was shown by Ruzsa in [Ruz84]. It is clear that (v) (vi), so we will now show that (vi) implies item (vi) of Theorem 5.1.2 in order to deduce that (vi) (i). To this end, we observe that if μ is a probability measure on T and we let $\nu = \frac{1}{2}(\mu + \tilde{\mu})$, then ν is a probability measure for which $\nu = \tilde{\nu}$. We see that μ is continuous if and only if ν is continuous, and that $\hat{\nu}(n) = \text{Re}(\hat{\mu}(n))$ for all n N, from which the desired result follows.

It is clear that (i) (iv) and (i) (ii) in light of Theorem 5.1.2, so we will now show that (ii) (v). Let μ be as in (v) and recall that $\hat{\mu}(n)$ R for all n N. By Lemma 5.2.1 let (X, B, ν, T) be an ergodic m.p.s. and f $L^2(X,\nu)$ a real-valued function for which $U^n f, f = \hat{\mu}(n)$ for all n N. By Birkhoff's ergodic theorem let X B be such that $\nu(X\ ) = 1$ and for every x X we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{n+k}x)\overline{f(T^n x)} = \int_X U^k f \overline{f} d\nu = \hat{\mu}(k) \tag{5.39}$$

for all $k \quad \mathbb{N}$, and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int_X f d\nu. \tag{5.40}$$

We see that for all $h \quad R$ and $x \quad X$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{n+h}x)\overline{f(T^n x)} = \hat{\mu}(h) = 0, \text{ hence} \tag{5.41}$$
$$\int_X f d\nu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = 0.$$

It now suffices to observe that

$$\begin{aligned} \mu(\{0\}) &= \int_{\mathsf{T}} \mathbb{1}_{\{0\}} d\mu = \int_{\mathsf{T}} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n x} d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_{\mathsf{T}} e^{2\pi i n x} d\mu \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \hat{\mu}(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \quad U^n f, f \quad = \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} U^n f, f \\ &= \quad \int_X f d\nu, f \quad = (\int_X f d\nu)^2 = 0. \end{aligned} \tag{5.42}$$

We will now show that (viii) (iii). Let $(x_n)_{n=1}^{\infty}$ be as in (iii) and let $k \quad \mathbb{N}$ be arbitrary. Let $y_n = (\cos(2\pi i k x_n), \sin(2\pi i k x_n))$ for all $n \quad \mathbb{N}$ and note that for each $h \quad R$ we have

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \quad y_{n+h}, y_n \\ =&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Big(\cos(2\pi i k x_{n+h}) \cos(2\pi i k x_n) + \sin(2\pi i k x_{n+h}) \sin(2\pi i k x_n) \Big) \\ =&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \cos(2\pi i k (x_{n+h} - x_n)) = 0, \end{aligned} \tag{5.43}$$

where the last equality follows from Theorem 4.2.8. We now see that

$$(0,0) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} y_n = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (\cos(2\pi i k x_n), \sin(2\pi i k x_n)),$$

so we have shown that $(x_n)_{n=1}^{\infty}$ is actually uniformly distributed.

We will show that (iii) (viii) by using ideas similar to those in [Ruz84]. Let $\mathsf{S}^1 = \{z \quad \mathsf{C} \mid |z| = 1\}$. Let us recall that R^2 as a real-Hilbert space is isomorphic to C under the inner product $\quad a, b \quad = \frac{1}{2}(a\overline{b} + \overline{a}b)$. Let $(y_n)_{n=1}^{\infty} \quad \mathsf{S}^1$ be a sequence for which

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{2}(y_{n+h}\overline{y_n} + \overline{y_{n+h}}y_n) = 0, \tag{5.44}$$

for every h R. Furthermore, we see that by replacing $(y_n)_{n=1}^{\infty}$ with $(cy_n)_{n=1}^{\infty}$ for some c C, we may assume without loss of generality that

$$\limsup_{N \to \infty} \frac{1}{N} | \sum_{n=1}^{N} y_n | = \limsup_{N \to \infty} \frac{1}{N} | \sum_{n=1}^{N} \mathrm{Re}(y_n) |. \tag{5.45}$$

Let $e : \mathsf{T} \to \mathsf{S}^1$ be the map given by $e(x) = \exp(2\pi i x)$. Since we will mostly be using exponential funtions in this proof, we remind the reader that $e(x) + e(-x) = \cos(2\pi i x)$. Let $(\xi_n)_{n=1}^{\infty}$ be a sequence of independent random variables from T to T, with $e(\xi_n)$ having density function $1 + \frac{1}{2}(y_n\overline{e(x)} + \overline{y_n}e(x))$. We see that for any n N, we have that

$$\int_{\mathsf{T}} e(\xi_n(x))dx = \int_{\mathsf{T}} (1 + \frac{1}{2}(y_n\overline{e(x)} + \overline{y_n}e(x)))e(x)dx = \frac{y_n}{2}. \tag{5.46}$$

We also see that for any n N, and j $\mathsf{Z} \setminus \{-1, 0, 1\}$, we have that

$$\int_{\mathsf{T}} e(j\xi_n(x)) = \int_{\mathsf{T}} (1 + \frac{1}{2}(y_n\overline{e(x)} + \overline{y_n}e(x)))e(x)^j dx = 0. \tag{5.47}$$

We now see from the strong law of large numbers, that for some B T of full Lebesgue measure, and every x B, we have that

$$\sum_{n=1}^{N} e(\xi_n(x)) = \frac{1}{2} \sum_{n=1}^{N} y_n + o(N), \text{ hence} \tag{5.48}$$
$$\overline{\lim_{N \to \infty}} \frac{1}{N} | \sum_{n=1}^{N} \frac{1}{2}(e(\xi_n(x)) + e(-\xi_n(x))) | = \frac{1}{2} \overline{\lim_{N \to \infty}} \frac{1}{N} | \sum_{n=1}^{N} \frac{1}{2}(y_n + \overline{y_n}) |$$

Furthermore, we see that for h H and j N, we have that

$$\int_{\mathsf{T}} \sum_{n=1}^{N} (e(j\xi_{n+h}(x) - j\xi_n(x)) + e(-j\xi_{n+h}(x) + j\xi_n(x)))dx \tag{5.49}$$
$$= \sum_{n=1}^{N} \Big((\int_{\mathsf{T}} e(j\xi_{n+h}(x))dx)(\int_{\mathsf{T}} e(-j\xi_n(x))dx) + (\int_{\mathsf{T}} e(-j\xi_{n+h}(x))dx)(\int_{\mathsf{T}} e(j\xi_n(x))dx) \Big)$$
$$= \begin{cases} \frac{1}{4} \sum_{n=1}^{N} (y_{n+h}\overline{y_n} + \overline{y_{n+h}}y_n) & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases} = o(N).$$

For each $j, n \quad \mathsf{N}$ and $h \quad H$, let $Y_{n,h,j}(x) = e(j\xi_{n+h}(x) - j\xi_n(x)) + e(-j\xi_{n+h}(x)^j + j\xi_n(x))$, and note that $\mathrm{Var}(Y_{j,n,h}) \leq 4$. We see that $Y_{j,n,h}$ and $Y_{j,m,h}$ are independent whenever $|m - n| > h$, so let $A_N = \{(n,m) \quad [1,N]^2 \mid |m-n| > h\}$, and note that $|A_N| \geq N(N-2h-1)$. We now see that for any $N \quad \mathsf{N}$, we have that

$$\int_{\mathsf{T}} |\sum_{n=1}^{N}(Y_{j,n,h}(x) - \int_{\mathsf{T}} Y_{j,n,h}(y)dy)|^2 dx \tag{5.50}$$
$$= \sum_{1 \leq n,m \leq N} \int_{\mathsf{T}} (Y_{j,n,h}(x) - \int_{\mathsf{T}} Y_{j,n,h}(y)dy)(\overline{Y_{j,m,h}(x)} - \int_{\mathsf{T}} \overline{Y_{j,m,h}(y)}dy)dx$$
$$\leq \left| \sum_{(n,m) \quad A_N} \left(\int_{\mathsf{T}} (Y_{j,n,h}(x) - \int_{\mathsf{T}} Y_{j,n,h}(y)dy)dx\right)\left(\int_{\mathsf{T}} (\overline{Y_{j,m,h}(x)} - \int_{\mathsf{T}} \overline{Y_{j,m,h}(y)}dy)dx\right)\right|$$
$$+ \sum_{(n,m) \quad A_N^c} 4$$
$$\leq 4(2h+1)N.$$

We now see from a standard Borel-Cantelli argument that for Lebesgue-a.e. $x \quad \mathsf{T}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} Y_{j,n,h}(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int_{\mathsf{T}} Y_{j,n,h}(y)dy = 0, \tag{5.51}$$

for every $j, n \quad \mathsf{N}$ and $h \quad H$. We now see that for some $A \quad \mathsf{T}$ of full Lebesgue measure, the sequence $(\xi_{n+h}(x) - \xi_n(x))_{n=1}^{\infty}$ is uniformly symmetrically distributed for every $h \quad H$, so $(\xi_n(x))_{n=1}^{\infty}$ is also uniformly symmetrically distributed for $x \quad A$. Letting $x \quad A \cap B$ be arbitrary, we now see that

$$\overline{\lim_{N\to\infty}} |\frac{1}{N}\sum_{n=1}^{N} y_n| = \overline{\lim_{N\to\infty}} |\frac{1}{N}\sum_{n=1}^{N}\mathrm{Re}(y_n)| = \overline{\lim_{N\to\infty}} |\frac{1}{N}\sum_{n=1}^{N}\frac{1}{2}(y_n + \overline{y_n})| \tag{5.52}$$
$$= \overline{\lim_{N\to\infty}} \frac{1}{N} |\sum_{n=1}^{N}(e(\xi_n(x)) + e(-\xi_n(x)))| = 0.$$

Lastly, we will show that (viii) (i) by showing that (viii) implies item (ii) of Theorem 5.1.2. Let $(x_n)_{n=1}^{\infty}$ be as in item (ii) of Theorem 5.1.2. We may view $(x_n)_{n=1}^{\infty}$ as a sequence of unit vectors in R^2 by identifying $a + bi$ with (a, b) and by observing that $x, y_{\ \mathsf{R}^2} = \frac{1}{2}(x\overline{y} + \overline{x}y) = \mathrm{Re}(x\overline{y})$. It now suffices to observe that for all $h \quad R$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_{n+h}, x_{n\ \mathsf{R}^2} = \mathrm{Re}(\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_{n+h}\overline{x_n}) = 0. \tag{5.53}$$

□

In order to show that every nice vdC set is also a set of nice recurrence, we first need to recall a classical result of Weyl ([Wey14], [Wey16]).

Theorem 5.2.3 (Theorem 1.4.1 in [KN74])**.** *If $(a_n)_{n=1}^{\infty}$ is a sequence of distinct integers, then $(a_n x)_{n=1}^{\infty}$ is uniformly distributed in $\mathbb{T}$ for a.e. $x \in \mathbb{T}$.*

Theorem 5.2.4. *Every nice vdC set is a set of nice recurrence.*

Proof. We will prove the contrapositive. Let $S \subseteq \mathbb{N}$ be a set that is not a set of nice recurrence. Choose a m.p.s. $(X, \mathcal{B}, \mu, T)$ and some $B \in \mathcal{B}$ for which

$$\mathcal{S} := \sup_{s \in S} \mu(B \cap T^{-s}B) < \mu(B)^2. \tag{5.54}$$

We will now use a standard argument to go from the setting of measure preserving systems to the setting of subsets of $\mathbb{N}$ with positive natural density. Let $x \in X$ be such that Birkhoff's Ergodic Theorem holds for all $f \in \{\mathbb{1}_{B \cap T^{-n}B}\}_{n \geq 0}$ when evaluated along the orbit of x and let $A := \{n \in \mathbb{N} \mid T^n x \in B\}$. We see that

$$d(A) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_B(T^n x) = \mu(B), \text{ and} \tag{5.55}$$

$$\begin{aligned} d(A \cap (A - m)) &= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_B(T^n x)\mathbb{1}_B(T^{n+m}x) \\ &= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{B \cap T^{-m}B}(T^n x) = \mu(B \cap T^{-m}B) \text{ for all } m \in \mathbb{N}. \end{aligned}$$

By Theorem 5.2.3, there exists $\alpha \in [0, 1]$ such that $(n^2\alpha)_{n \in A^c}, (-n^2\alpha)_{n \in A^c \cap (A-s)}, ((n + s)^2\alpha)_{n \in A \cap (A^c - s)}, ((2ns + s^2)\alpha)_{n \in A^c \cap (A^c - s)}$ are all uniformly distributed in $\mathbb{T}$, for every $s \in S$, whenever they are infinite sequences. Let us now consider the sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{T}$ given below.

$$x_n = \begin{cases} 0 & \text{if } n \in A \\ n^2\alpha & \text{if } n \in A^c \end{cases} \tag{5.56}$$

We now see that for any $s \in S$ we have that

$$x_{n+s} - x_n = \begin{cases} 0 & \text{if } n \in A \cap (A - s) \\ -n^2\alpha & \text{if } n \in (A - s) \cap A^c \\ (n + s)^2\alpha & \text{if } n \in A \cap (A^c - s) \\ (2ns + s^2)\alpha & \text{if } n \in A^c \cap (A^c - s) \end{cases}. \tag{5.57}$$

Let $k \in \mathbb{N}$ be arbitrary, and let $y_n = \exp(2\pi i k x_n)$ for all $n \in \mathbb{N}$. We use the fact that $(x_{n+s} - x_n)_{n \in (A-s)\cap A^c}, (x_{n+s} - x_n)_{n \in A\cap(A^c-s)}, (x_{n+s} - x_n)_{n \in A^c\cap(A^c-s)}$ are all uniformly distributed if they are infinite sequences to obtain equation (5.59) in the calculations below.

$$\lim_{N\to\infty} \Big|\frac{1}{N}\sum_{n=1}^{N} y_{n+s}\overline{y_n}\Big| \tag{5.58}$$
$$= \lim_{N\to\infty} \frac{1}{N}\Big| \sum_{n \in A\cap(A-s)\cap[1,N]} y_{n+s}\overline{y_n} + \sum_{n \in (A-s)\cap A^c\cap[1,N]} y_{n+s}\overline{y_n}$$
$$+ \sum_{n \in A\cap(A^c-s)\cap[1,N]} y_{n+s}\overline{y_n} + \sum_{n \in A^c\cap(A^c-s)\cap[1,N]} y_{n+s}\overline{y_n}\Big|$$
$$\leq \lim_{N\to\infty} \Big|\frac{1}{N} \sum_{n \in A\cap(A-s)\cap[1,N]} \exp(2\pi i k(x_{n+s} - x_n))\Big|$$
$$+ \lim_{N\to\infty} \Big|\frac{1}{N} \sum_{n \in (A-s)\cap A^c\cap[1,N]} \exp(2\pi i k(x_{n+s} - x_n))\Big|$$
$$+ \lim_{N\to\infty} \Big|\frac{1}{N} \sum_{n \in A\cap(A^c-s)\cap[1,N]} \exp(2\pi i k(x_{n+s} - x_n))\Big|$$
$$+ \lim_{N\to\infty} \Big|\frac{1}{N} \sum_{n \in A^c\cap(A^c-s)\cap[1,N]} \exp(2\pi i k(x_{n+s} - x_n))\Big|$$
$$= \lim_{N\to\infty} \Big|\frac{1}{N} \sum_{n \in A\cap(A-s)} 1\Big| = \overline{d}(A\cap(A-s)) \leq S. \tag{5.59}$$

Since $(x_n)_{n \in A^c}$ is uniformly distributed in $\mathbb{T}$ we see that

$$\limsup_{\substack{s\to\infty \\ s \in S}} \lim_{N\to\infty} \Big|\frac{1}{N}\sum_{n=1}^{N} y_{n+s}\overline{y_n}\Big| \leq S < \overline{d}(A)^2 \tag{5.60}$$
$$= \Big(\lim_{N\to\infty}\frac{1}{N}\sum_{n \in A\cap[1,N]} 1\Big)^2 = \Big|\lim_{N\to\infty}\frac{1}{N}\sum_{n \in A\cap[1,N]} y_n + \lim_{N\to\infty}\frac{1}{N}\sum_{n \in A^c\cap[1,N]} y_n\Big|^2$$
$$= \lim_{N\to\infty}\Big|\frac{1}{N}\sum_{n=1}^{N} y_n\Big|^2,$$

so S is not a nice vdC set. □

Theorem 5.2.5. *For $R \subseteq \mathbb{N}$ the following are equivalent.*

(i) R is a nice FC$^+$ set.

(ii) R is a nice vdC set.

(iii) For every Hilbert space $\mathcal{H}$, every unitary operator $U : \mathcal{H} \to \mathcal{H}$, and every $f \in \mathcal{H}$, we have

$$\sup_{r \in R} |\langle U^r f, f \rangle| \geq ||Pf||^2, \tag{5.61}$$

where P is the projection onto the subspace of U-invariant elements.

(iv) For any positive measure μ on T we have

$$\sup_{r \in R} |\hat{\mu}(r)|^2 \geq \sum_{t \in A} \mu(\{t\})^2, \tag{5.62}$$

where $A \subseteq \mathsf{T}$ is the set of atoms of μ.

(v) For every Hilbert space H, every unitary operator $U : H \to H$, and every $f \in H$, we have

$$\sup_{r \in R} |\langle U^r f, f \rangle| \geq ||P_c f||^2, \tag{5.63}$$

where $H = H_c \oplus H_{wm}$ is the Jacobs-de Leeuw-Glicksberg decomposition (cf. Theorem 2.3.5) and P_c is the orthogonal projection of H onto H_c.

Proof. In [BL08] it is shown that (i) $\Rightarrow$ (ii), so let us now show that (ii) $\Rightarrow$ (iii). By the spectral theorem (Theorem B.4 in [EW11]) let μ be a positive measure on T for which $\hat{\mu}(n) = \langle U^n f, f \rangle$ for all $n \in \mathsf{N}$ and $\mu(\{0\}) = ||Pf||^2$ where P is the orthogonal projection onto the space of U invariant vectors. By Lemma 5.2.1 there exists an ergodic m.p.s. $(X, \mathcal{B}, \nu, T)$ and (by abuse of notation) $f \in L^2(X, \nu)$ for which $\langle U^n f, f \rangle = \hat{\mu}(n)$ for all $n \in \mathsf{N}$ and $\mu(\{0\}) = ||Pf||^2 = (\int_X f d\nu)^2$. By Birkoff's ergodic theorem we see that for a.e. $x \in \mathsf{T}$ we have

$$\langle U^h f, f \rangle = \int_{\mathsf{T}} U^h f \overline{f} d\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{n+h}x)\overline{f(T^n x)} \text{ for all } h \in \mathsf{N}, \text{ and} \tag{5.64}$$

$$\int_{\mathsf{T}} f d\nu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x),$$

from which the desired result is immediate.

It is clear that (iv) $\Rightarrow$ (i), so we will now show that (i) $\Rightarrow$ (iv). Let μ be a positive measure on T, let $A \subseteq \mathsf{T}$ denote the set of atoms of μ, and let $\nu = \mu * \tilde{\mu}$. We see that

$$\nu(\{0\}) = \sum_{t_1+t_2=0} \mu(\{t_1\})\tilde{\mu}(\{t_2\}) = \sum_{t \in A} \mu(\{t\})\mu(-\{-t\}) = \sum_{t \in A} \mu(\{t\})^2, \text{ and} \tag{5.65}$$

$$\hat{\nu}(n) = \hat{\mu}(n)\hat{\tilde{\mu}}(n) = |\hat{\mu}(n)|^2 \text{ for all } n \in \mathsf{N}.$$

Since R is a nice FC^+ set we see that

$$\sup_{r\in R}|\hat{\mu}(r)|^2 = \sup_{r\in R}|\hat{\nu}(r)| \geq \nu(\{0\}) = \sum_{t\in A}\mu(\{t\})^2. \tag{5.66}$$

The fact that (iii) $\Rightarrow$ (i) is an immediate consequence of Lemma 6 of [NRS12]. It is clear that (v) $\Rightarrow$ (iii), so we will now show that (i) $\Rightarrow$ (v). By the spectral theorem (Theorem B.4 in [EW11]) let μ be a positive measure on $\mathbb{T}$ for which there exists an isomorphism $S : L^2([0,1],\mu) \to H$ satisfying $S(e^{2\pi inx}) = U^n f$ for all $n \in \mathbb{Z}$. Let $A \subseteq [0,1]$ denote the set of atoms of μ, and for each $t \in [0,1]$ let P_t denote the orthogonal projection of H onto the space of eigenfunctions with eigenvalue $e^{2\pi it}$. We observe that $||P_t f||^2 = \mu(\{t\})$ for all $t \in [0,1]$. It now suffices to observe that

$$\begin{aligned}\sup_{r\in R}|\langle U^r f, f\rangle| &= \sup_{r\in R}\left|\int_0^1 e^{2\pi irx}d\mu(x)\right| = \sup_{r\in R}|\hat{\mu}(r)| \geq \sqrt{\sum_{t\in A}\mu(\{t\})^2} = \sqrt{\sum_{t\in A}||P_t f||^4} \\ &\geq \sum_{t\in A}||P_t f||^2 = ||P_c f||^2.\end{aligned} \tag{5.67}$$

□

Theorem 5.2.6. *Let $R \subseteq \mathbb{N}$, H be a Hilbert space, and $(y_n)_{n=1}^{\infty} \subseteq SA(H)$.*

(i) If R is a vdC set and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle y_{n+r}, y_n\rangle = 0 \tag{5.68}$$

for all $r \in R$, then $(y_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(ii) If R is an enhanced vdC set and

$$\lim_{\substack{r\to\infty \\ r\in R}}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle y_{n+r}, y_n\rangle\right| = 0, \tag{5.69}$$

then $(y_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(iii) If $R = (r_n)_{n=1}^{\infty}$ is a density vdC set and

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle y_{n+r_m}, y_n\rangle\right| = 0, \tag{5.70}$$

then $(y_n)_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(iv) In (ii) and (iii) is is possible for $(y_n)_{n=1}^{\infty}$ to be a rigid sequence. In particular, $(y_n)_{n=1}^{\infty}$ need not be a nearly mildly mixing sequence.

Proof. We will first prove items (i)-(iii). Let $(N_q)_{q=1}^{\infty}$ be any sequence for which

$$\gamma(h) := \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} y_{n+h}, y_n \tag{5.71}$$

exists for all h N. Since $(\gamma(h))_{h=1}^{\infty}$ is a positive definite sequence, by Bochner's Theorem let μ be a positive measure on T for which $\hat{\mu}(h) = \gamma(h)$ for all h N. In each of (i)-(iii) we conclude that μ is a continuous measure by using Theorem 5.1.2(vi) for part (i), Theorem 5.2.7(i) for (ii), and Theorem 5.2.7(ii) for (iii). Since μ is continuous we may use Wiener's Theorem to see that

$$0 = \lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} |\hat{\mu}(h)|^2 \geq \left(\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} |\hat{\mu}(h)|\right)^2 \tag{5.72}$$
$$= \lim_{H\to\infty} \frac{1}{H} \left| \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} y_{n+h}, y_n \right|,$$

so Theorem 2.2.10 tells us that $(y_n)_{n=1}^{\infty}$ i a nearly weakly mixing sequence.

While we can give a direct proof of (iv), it would be much more cumbersome than a direct proof of Theorem 5.1.10(iv) (Theorem 5.2.9(iv)) which we do provide. Consequently, we will deduce part (iv) from Theorem 5.1.10(iv). To this end, it suffices to observe that if $((y_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, $H = H((y_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$, and $S : H \to H$ is the unitary operator induced by the left shift, then $(y_n)_{n=1}^{\infty}$ is a mildly mixing (rigid) element of (H, S) for all $(N_q)_{q=1}^{\infty}$ N if and only if $(y_n)_{n=1}^{\infty}$ is a nearly mildly mixing (rigid) sequence. □

Before proving our next main theorem, we require the following preliminary results.

Theorem 5.2.7 (cf. Theorems 2.1 and 3.6 in [BL08])**.** *Let R N.*

(i) R is an enhanced vdC set if and only if it is a FC$^+$ set.

(ii) R is a density vdC set if and only if it is a density FC$^+$ set.

Lemma 5.2.8. *Let R N.*

(i) If $d(R) = 1$ then R is a density vdC set.

(ii) If for each D N there exists $r_1 < r_2 < \cdots < r_D$ N for which $r_n - r_m$ R for all $1 \leq m < n \leq D$, then R is a nice vdC set.

Proof of (i). Due to Theorem 5.2.7(ii) it suffices to show that R is a density FC$^+$ set. For $t \in \mathbb{T}$ let μ_t be the measure given by $\mu_t(E) = \mu(E+t)$ and recall that $\hat{\mu}_t(n) = e^{-2\pi int}\hat{\mu}(n)$. It now suffices to observe that for $R = (r_m)_{m=1}^{\infty}$ with $d(R) = 1$ and $t \in \mathbb{T}$ we have

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}|\hat{\mu}_t(r_m)| = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\hat{\mu}_t(n)| \geq \lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\hat{\mu}_t(n)\right| \tag{5.73}$$
$$= \lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\int_{\mathbb{T}}e^{-2\pi inx}d\mu_t(x)\right| = \left|\int_{\mathbb{T}}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{-2\pi inx}d\mu_t(x)\right|$$
$$= \left|\int_{\mathbb{T}}\mathbb{1}_{\{0\}}(x)d\mu_t(x)\right| = \mu_t(\{0\}) = \mu(\{t\}).$$

□

Proof of (ii). We remark that our proof is very similar to the classical proof of Theorem 2.1.2(i). Let $(u_n)_{n=1}^{\infty}$ be a sequence of complex numbers of norm 1, and let $\epsilon > 0$ be arbitrary. Let $D \in \mathbb{N}$ be such that $\frac{1}{D-1} < \epsilon$, and let $r_1 < r_2 < \cdots < r_D \in \mathbb{N}$ be such that $r_n - r_m \in R$ for all $1 \leq m < n \leq D$. We now see that

$$\limsup_{N\to\infty}|\frac{1}{N}\sum_{n=1}^{N}u_n|^2 = \limsup_{N\to\infty}|\frac{1}{D}\sum_{d=1}^{D}\frac{1}{N}\sum_{n=1}^{N}u_{n+r_d}|^2 \tag{5.74}$$
$$\leq \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\frac{1}{D}\sum_{d=1}^{D}u_{n+r_d}|^2 = \limsup_{N\to\infty}\frac{1}{ND^2}\sum_{n=1}^{N}\sum_{1\leq d_1,d_2\leq D}u_{n+r_{d_1}}\overline{u_{n+r_{d_2}}}$$
$$\leq \frac{1}{D} + \sum_{\substack{1\leq d_1,d_2\leq D\\ d_1 = d_2}}\limsup_{N\to\infty}|\frac{1}{N}\sum_{n=1}^{N}u_{n+r_{d_1}}\overline{u_{n+r_{d_2}}}|.$$

Letting

$$M = \max_{\substack{1\leq d_1,d_2\leq D\\ d_1 = d_2}}\limsup_{N\to\infty}|\frac{1}{N}\sum_{n=1}^{N}u_{n+r_{d_1}}\overline{u_{n+r_{d_2}}}| \tag{5.75}$$
$$= \max_{\substack{1\leq d_1,d_2\leq D\\ d_1 = d_2}}\limsup_{N\to\infty}|\frac{1}{N}\sum_{n=1}^{N}u_{n+r_{d_1}-r_{d_2}}\overline{u_n}|,$$

we conclude from (5.74) that

$$\limsup_{N\to\infty}|\frac{1}{N}\sum_{n=1}^{N}u_n|^2 \leq \frac{1}{D} + M\frac{D-1}{D}, \text{ hence} \tag{5.76}$$
$$M \geq \frac{D}{D-1}\limsup_{N\to\infty}|\frac{1}{N}\sum_{n=1}^{N}u_n|^2 - \frac{1}{D-1} \geq \limsup_{N\to\infty}|\frac{1}{N}\sum_{n=1}^{N}u_n|^2 - \epsilon.$$

$\square$

Theorem 5.2.9. *Let $R \subseteq \mathbb{N}$, $\mathcal{H}$ be a Hilbert space, $U : \mathcal{H} \to \mathcal{H}$ a unitary operator, and let $x \in \mathcal{H}$. Let $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{wm}$ be the Jacobs-de Leeuw-Glicksberg Decomposition and let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_c$.*

(i) If R is a vdC set and $\langle U^r x, x\rangle = 0$ for all $x \in R$, then $Px = 0$.

(ii) If R is an enhanced vdC set and

$$\lim_{\substack{r\to\infty \\ r \in R}} \langle U^r x, x\rangle = 0, \tag{5.77}$$

then $Px = 0$.

(iii) If $R = (r_n)_{n=1}^{\infty}$ is a density vdC set and

$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \langle U^{r_m} x, x\rangle = 0, \tag{5.78}$$

then $Px = 0$.

Proof. We will first prove items (i)-(iii). Since $(\langle U^{-h}x, x\rangle)_{h=1}^{\infty}$ is a positive definite sequence, by Bochner's Theorem let μ be a positive measure on $\mathbb{T}$ for which $\hat{\mu}(h) = \langle U^{-h}x, x\rangle$ for all $h \in \mathbb{N}$. In each of (i)-(iii) we conclude that μ is a continuous measure by using Theorem 5.1.2(vi) for part (i), Theorem 5.2.7(i) for (ii), and Theorem 5.2.7(ii) for (iii). Since μ is continuous we may use Wiener's Theorem to see that

$$0 = \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} |\hat{\mu}(h)|^2 \geq \left(\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} |\hat{\mu}(h)|\right)^2 = \lim_{H\to\infty} \left|\frac{1}{H}\sum_{h=1}^{H} \langle U^{-h}x, x\rangle\right|^2, \tag{5.79}$$

so $x \in \mathcal{H}_{\text{wm}}$.

We can now prove item (iv). Let $\mathbf{X} := (X, \mathcal{B}, \mu, T)$ be any weakly mixing m.p.s. that is not mildly mixing and let $f \in L^2(X, \mu)$ be any rigid function satisfying $\int_X f d\mu = 0$. The assumption that f is rigid is only for the sake of concreteness since our argument can be applied to any weakly mixing element of $(L^2(X, \mu), U_T)$. Since $\mathbf{X}$ is weakly mixing, we see that the equation in item (iii) is satisfied when $R = \mathbb{N}$, and $\mathbb{N}$ is a density vdC set by Lemma 5.2.8(i), so it only remains to verify that item (iv) holds for (ii). Since $\mathbf{X}$ is weak mixing, we see that for each $k \in \mathbb{N}$ the sets

$$R_k := \{n \in \mathbb{N} \mid |\langle U^n f, f\rangle| < \frac{1}{k}\} \tag{5.80}$$

satisfy $d(R_m) = 1$. We will now inductively construct a sequence $(n_k)_{k=1}^{\infty}$ for which $n_{k_1} - n_{k_2} \in B^c$ for all $k_1 > k_2$. For the base case let $n_1 \in R_1$ be arbitrary. For the inductive step, assume that $n_1, \cdots, n_k$ have been chosen, and note that $d(R_k + n_j) = 1$ for all $1 \leq j \leq k$, so for

$$B_k := R_k \cap \bigcap_{j=1}^{k}(R_k + n_j)), \tag{5.81}$$

we have $d(B_k) = 1$. Since $B_k \neq \emptyset$, let $n_{k+1} \in B_k$ be arbitrary, and note that for any $1 \leq j \leq k$ we have $n_{k+1} - n_j \in R_k$. We see that $R = \{n_{k_1} - n_{k_2} \mid k_1 > k_2 \geq 1\}$ is a nice vdC set by Lemma 5.2.8(ii). We recall that every nice vdC set is also an enhanced vdC set. It now suffices to observe that for $k_1 > k_2 \geq 1$ we have $|\langle U^{n_{k_1} - n_{k_2}} f, f\rangle| < \frac{1}{k_1 - 1}$, so the equation in item (ii) is satisfied for the R that we have constructed. □

Theorem 5.2.10. *Let $R \subseteq \mathbb{N}$, $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$, and let $\overline{D}$ be the measure of discrepancy discussed in Definition 2.4.3.*

(i) If R is a vdC set and $(x_{n+r} - x_n)_{n=1}^{\infty}$ is uniformly distributed for all $r \in R$, then $(x_n)_{n=1}^{\infty}$ is a wm-sequence.

(ii) If R is an enhanced vdC set and

$$\lim_{\substack{r \to \infty \\ r \in R}} \overline{D}((x_{n+r} - x_n)_{n=1}^{\infty}) = 0, \tag{5.82}$$

then $(x_n)_{n=1}^{\infty}$ is a wm-sequence.

(iii) If $R = (r_n)_{n=1}^{\infty}$ is a density vdC set and

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \overline{D}((x_{n+r_m} - x_n)_{n=1}^{\infty}) = 0, \tag{5.83}$$

then $(x_n)_{n=1}^{\infty}$ is a wm-sequence.

(iv) In (ii) and (iii) $(x_n)_{n=1}^{\infty}$ need not be an mm-sequence. In fact, it is possible to have $(f(x_n))_{n=1}^{\infty}$ be a rigid sequence for all $f \in C([0, 1])$ satisfying $\int_0^1 f(x)dx = 0$.

Proof of (i). For each $k \in \mathbb{N}$ and $r \in R$, we see that

$$0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k(x_{n+r}-x_n)} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_{n+r}} \overline{e^{2\pi i k x_n}}, \tag{5.84}$$

so by Theorem 5.2.6(i) we see that $(e^{2\pi i k x_n})_{n=1}^{\infty}$ is a nearly weakly mixing sequence. Since $k \in \mathbb{N}$ was arbitrary, we see that $(x_n)_{n=1}^{\infty}$ is a wm-sequence. □

Proof of (ii) and (iii). Let $\epsilon > 0$ be arbitrary and let $\gamma_r = \overline{D}((x_{n+r} - x_n)_{n=1}^{\infty})$. We argue as we did in the proof of (iv) $\Rightarrow$ (i) in Theorem 2.4.17 to see that for all $k \in \mathbb{N}$ and $r \in R$ we have

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k(x_{n+r}-x_n)} \le \epsilon + c(k,\epsilon)\gamma_r. \tag{5.85}$$

Depending on whether we are proving (ii) or (iii), we see that

$$\lim_{\substack{r\to\infty \\ r\in R}} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_{n+r}} \overline{e^{2\pi i k x_n}} \right| \le \epsilon, \text{ or} \tag{5.86}$$

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_{n+r_m}} \overline{e^{2\pi i k x_n}} \right| \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we may apply Theorem 5.2.6(ii)-(iii) to see that $(e^{2\pi i k x_n})_{n=1}^{\infty}$ is a nearly weakly mixing sequence. Since $k \in \mathbb{N}$ was arbitrary, we see that $(x_n)_{n=1}^{\infty}$ is a wm-sequence. □

We suspect that in items (ii) and (iii) of Theorem 5.2.10 that the sequence $(x_n)_{n=1}^{\infty}$ need not be a mm-sequence, and may in fact be such that $(f(x_n))_{n=1}^{\infty}$ is a rigid sequence for any $f \in C([0,1])$ satisfying $\int_0^1 f(x)dx = 0$. We would like to show this by taking a weakly mixing m.p.s. $(X, \mathcal{B}, \mu, T)$ that also has a nontrivial rigid factor, and setting $x_n = T^n x$ for some generic $x \in X$. The reason that we are currently unable to make use of these ideas is due to technicalities that are hinted at by Question 3.4.5.

CHAPTER 6

ON THE PARTITION REGULARITY OF $ax + by = cw^m z^n$

This chapter is the result of a collaboration with Richard Magner during the last year of his Ph.D. at Boston University. I had obtained many of the results in sections 3, 5, 8, and 9 on my own, but the results of section 4 that I knew I needed were out of my reach. Consequently, I contacted Richard for help since he is a specialist in the area. Our discussions also gave rise to sections 6, 7, and some improvements to section 8. At the time of the writing of this thesis this chapter has been submitted for publication. We ask the reader who wishes to cite results from this chapter of the thesis to cite them from the published version so that Richard Magner may also receive credit for his contributions.

6.1 Introduction

We say that an equation is *partition regular over* S if for any finite partition $S = \bigcup_{i=1}^{r} C_i$, there exists some C_i containing a solution to the equation. One of the first results about partition regular diophantine equations is the celebrated theorem of Schur ([Sch16]), which established the partition regularity of $x + y = z$ over $\mathbb{N}$. Schur's student Rado ([Rad33]) classified which finite systems of linear homogeneous equations are partition regular over $\mathbb{N}$. Since it is not known whether or not the equation $x^2 + y^2 = z^2$ is partition regular over $\mathbb{N}$, we are still far from achieving a classification for which systems of polynomial equations are partition regular. One of the first results in this direction is a theorem of Bergelson ([Ber96], page 53), which shows that the equation $x - y = p(z)$ is partition regular over $\mathbb{N}$ for any polynomial $p(x) \in \mathbb{Z}[x]$ which satisfies $p(0) = 0$. While the result of Bergelson shows that the equation $x - y = z^2$ is partition regular over $\mathbb{N}$, Csikvári, Gyarmati and Sárkozy ([CGS12]) showed that the equation $x + y = z^2$ is *not* partition regular over $\mathbb{N}$, and asked whether the equation $x+y = wz$ is partition regular over $\mathbb{N}$. Their question was answered in the positive, independently by Bergelson ([Ber10], Section 6) and Hindman ([Hin11]). Both proofs make use of ultrafilters and the algebra of the Stone-Čech compactifation. More examples of partition regular polynomial equations can be found in [DNLB18]. Results regarding necessary conditions for a polynomial equation to be partition regular over $\mathbb{N}$ can be found in [BLM21] and [DNLB18].

The polynomial equations mentioned so far have a simple form, but the proofs of their partition regularity properties are quite specific to these cases. Since there is currently no unified theory for the partition regularity of polynomial equations, any class of equations whose partition regularity is known may provide insight towards such a theory. In this paper, we give a partial classification of the partition regularity of equations of the form $ax + by = cw^m z^n$, where a, b, c $\mathbb{Z} \setminus \{0\}$ and m, n $\mathbb{N}$ are parameters, and x, y, z, w are variables. Theorem 6.1.1 is the main result of this paper. Before stating Theorem 6.1.1, we note that we remove 0 when considering partition regularity over a ring R in order to avoid trivial solutions. We also recall that any equation which is partition regular over a set S_1 (such as $\mathbb{N}$) is also partition regular over any set S_2 that contains S_1 (such as $\mathbb{Z} \setminus \{0\}$), but the converse is not true in general.

Theorem 6.1.1. *Fix* a, b, c $\mathbb{Z} \setminus \{0\}$ *and* m, n $\mathbb{N}$.

(i) Suppose that $m, n \geq 2$.

(a) If $a + b = 0$, *then the equation*

$$ax + by = cw^m z^n \tag{6.1}$$

is not partition regular over $\mathbb{Z} \setminus \{0\}$.

(b) If $a + b = 0$, *then equation* (6.1) *is partition regular over* $\mathbb{N}$.

(ii) If one of $\frac{a}{c}, \frac{b}{c}$, *or* $\frac{a+b}{c}$ *is an* n*th power in* $\mathbb{Q}$, *then the equation*

$$ax + by = cwz^n \tag{6.2}$$

is partition regular over $\mathbb{Z} \setminus \{0\}$. *If one of* $\frac{a}{c}, \frac{b}{c}$, *or* $\frac{a+b}{c}$ *is an* n*th power in* $\mathbb{Q}_{\geq 0}$, *then equation* (6.2) *is partition regular over* $\mathbb{N}$.

(iii) Assume that equation (6.2) *is partition regular over* $\mathbb{Q} \setminus \{0\}$.[1]

(a) If n *is odd, then one of* $\frac{a}{c}, \frac{b}{c}$, *or* $\frac{a+b}{c}$ *is an* n*th power in* $\mathbb{Q}$.

(b) If $n = 4, 8$ *is even, then one of* $\frac{a}{c}, \frac{b}{c}$, *or* $\frac{a+b}{c}$ *is an* $\frac{n}{2}$*th power in* $\mathbb{Q}$.

(c) If n *is even, then either one of* $\frac{a}{c}, \frac{b}{c}$, *or* $\frac{a+b}{c}$ *is a square in* $\mathbb{Q}$, *or* $(\frac{a}{c})(\frac{b}{c})(\frac{a+b}{c})$ *is a square in* $\mathbb{Q}$.

[1] To give a converse to part (ii) of Theorem 6.1.1 we would assume that equation (6.2) is partition regular over $\mathbb{N}$ or $\mathbb{Z} \setminus \{0\}$. The assumption that equation (6.2) is partition regular over $\mathbb{Q} \setminus \{0\}$ is weaker than either of the previous assumptions since $\mathbb{N}$ $\mathbb{Z} \setminus \{0\}$ $\mathbb{Q} \setminus \{0\}$.

To prove item (i)(a) we use one of the Rado conditions for polynomial equations that is proven in [BLM21], and we will show that item (i)(b) is an easy consequence of the partition regularity of the equation $x - y = z^n$. To prove item (ii) we use ultrafilters similar to those that were used in [Ber10] and [Hin11]. To prove item (iii) we use the classic partitions that were used by Rado in [Rad33] when determining necessary conditions for a finite system of linear homogeneous equations to be partition regular. In order to demonstrate that these classic partitions yield our desired results, we are required to prove Theorem 6.1.2, which is a partial generalization of the criteria of Grunwald and Wang for when $\alpha \in \mathbb{Z}$ is an nth power modulo every prime $p \in \mathbb{N}$ ([Gru33],[Wan48],[Wan50]). In order to state Theorem 6.1.2 we define the following notation. If $p \in \mathbb{N}$ is a prime and $r, s \in \mathbb{Z}$ are such that $p \nmid s$, then $\frac{r}{s} \equiv rs^{-1} \pmod{p}$.

Theorem 6.1.2 (cf. Corollary 6.4.2)**.** *Let* $\alpha, \beta, \gamma \in \mathbb{Q} \setminus \{0\}$.

(i) Suppose n *is odd and* α, β, γ *are not* n*th powers; or*

(ii) Suppose n *is even,* α, β, γ *are not* $\frac{n}{2}$*th powers, and* α *is not an* $\frac{n}{4}$*th power if* $4 \mid n$.

Then there exists infinitely many primes $p \in \mathbb{N}$ *modulo which none of* α, β, γ *are* n*th powers.*

We remark that we will apply Theorem 6.1.2 to $\alpha = \frac{a}{c}, \beta = \frac{b}{c}$, and $\gamma = \frac{a+b}{c}$. Since $\alpha = \beta + \gamma$, the condition that at least one of α, β, and γ not be an $\frac{n}{4}$th power if $4 \mid n$ is automatic by Fermat's Last Theorem when $n > 8$. We would also like to apologize to the reader for using p to denote ultrafilters, generic polynomials, and primes in $\mathbb{N}$. Thankfully, we do not have any proofs or statements that simultaneously make use of an ultrafilter, a generic polynomial, and/or a prime, so the meaning of p will be clear from the context.

The structure of the paper is as follows. In Section 6.2 we provide a statement of Rado's Theorem and briefly review some facts about the usage of ultrafilters in Ramsey Theory. A major goal of this section is to quickly familiarize the reader who is inexperienced with Ramsey Theory with enough basic knowledge of ultrafilters that they will be able to use as a blackbox the special kinds of ultrafilters introduced in Theorems 6.2.8 and 6.6.1. In Section 6.3 we prove items (i)(b), (ii), and (iii)(a-b) of Theorem 6.1.1 as Corollary 6.3.2, Theorem 6.3.5 and Corollary 6.3.10 respectively.

The main result of Section 6.4 is Theorem 6.1.2. Since our proof of Theorem 6.1.2 already requires us to work in finite extensions of $\mathbb{Q}$, we also prove a similar result as Lemma 6.4.7 in the more general setting of rings of integers of number fields. We then prove item (iii)(c) of Theorem 6.1.1 as Corollary 6.4.9, and we conclude the section with Lemma 6.4.10 which is an analogue of Theorem 6.1.2 for 2 variables.

In Section 6.5 we prove item (i)(a) of Theorem 6.1.1 as Theorem 6.5.1. We also determine the partition regularity of some equations of the form $ax + by = cwz^n$ which are not already addressed by Theorem 6.1.1 through the use of Lemma 6.5.14. In Section 6.6 we investigate the partition regularity of $ax + by = cwz^n$ over general integral domains R and attain results analogous to Theorem 6.1.1 while using methods very similar to those used in Section 6.3. In Section 6.7 we investigate the partition regularity of systems of equations of the form $a_ix_i + b_iy_i = c_iw_iz_i^n, 1 \le i \le k$ over integral domains R and attain results that support Conjecture 6.1.3.

Conjecture 6.1.3 (cf. Conjecture 6.8.6)**.** *Let* $a_1, \cdots, a_k, b_1, \cdots, b_k, c_1, \cdots, c_k \in \mathbb{Z} \setminus \{0\}$ *and* $n \in \mathbb{N}$*. The system of equations*

$$\begin{array}{ccccc} a_1x_1 & + & b_1y_1 & = & c_1w_1z_1^n \\ a_2x_2 & + & b_2y_2 & = & c_2w_2z_2^n \\ & & \vdots & & \\ a_kx_k & + & b_ky_k & = & c_kw_kz_k^n \end{array} \tag{6.3}$$

is partition regular over $\mathbb{Z} \setminus \{0\}$ *if and only if*

$$I := \bigcap_{i=1}^{k} \left\{ \frac{a_i}{c_i}, \frac{b_i}{c_i}, \frac{a_i + b_i}{c_i} \right\} \tag{6.4}$$

contains an n*th power in* $\mathbb{Q}$*.*

In Section 6.8 we state some conjectures and examine some equations and systems of equations whose partition regularity remains unknown. We also elaborate on the distinction between partition regularity of a polynomial equation over $\mathbb{N}$ instead of $\mathbb{Z} \setminus \{0\}$ by considering some illustrative examples of polynomial equations over $\mathbb{Z}[\sqrt{2}]$.

The main purposes of Section 6.9 is to give a thorough proof of the existence of the ultrafilters in Theorems 6.2.8 and 6.6.1 as Theorem 6.9.12 and 6.9.18 respectively. To this end, we begin the section with a detailed introduction to the theory of ultrafilters and its applications to semigroup theory. While the ultrafilter that we use in Theorem 6.9.12 is the same as the ultrafilter used in [Ber10] and [Hin11] (cf. Remark 6.9.13), an analogous ultrafilter need not exist over a general integral domain R. Consequently, we prove Corollary 6.9.26 and Theorem 6.9.28 to obtain a characterization of the integral domains R which possess an ultrafilters analogous to the one in Theorem 6.9.12.

6.2 Preliminaries

In this section we review some facts and useful tools in the study of partition regular equations.

Definition 6.2.1. *Given a set S, a ring R, and functions $f_1, \cdots, f_k : S^n \to R$, the system of equations*

$$\begin{array}{rcl} f_1(x_1, \cdots, x_n) & = & 0 \\ f_2(x_1, \cdots, c_n) & = & 0 \\ \vdots & \vdots & \vdots \\ f_k(x_1, \cdots, x_n) & = & 0 \end{array} \tag{6.5}$$

*is **partition regular over S** if for any finite partition $S = \bigcup_{i=1}^{r} C_i$, there exists $1 \le i_0 \le r$ and $x_1, \cdots, x_n \quad C_{i_0}$ which satisfy (6.5). If the set S is understood from context, then we simply say that the system of equations is **partition regular**.*

Definition 6.2.2. *Let R be an integral domain with field of fractions K. A matrix $\boldsymbol{M}$ $\mathrm{M}_{m \times n}(R)$ satisfies the **columns condition** if there exists a partition $C_1, \cdots, C_k$ of the column indices such that for $\vec{s}_i = \sum_{j \quad C_i} \vec{c}_j$ we have*

(i) $\vec{s}_1 = (0, \cdots, 0)^T$.

(ii) For all $i \ge 2$, we have

$$\vec{s}_i \quad \mathit{Span}_K\{\vec{c}_j \mid j \quad C_\ell, 1 \le \ell < i\}. \tag{6.6}$$

The columns condition was used by Rado to classify when a finite system of homogeneous linear equations is partition regular.

Theorem 6.2.3 (Rado, [Rad33])**.** *Given $\boldsymbol{M} \quad \mathrm{M}_{m \times n}(\mathsf{Z})$, the system of equations*

$$\boldsymbol{M}(x_1, \cdots, x_n)^T = 0 \tag{6.7}$$

is partition regular over N if and only if $\boldsymbol{M}$ satisfies the columns condition.

Corollary 6.2.4. *For $a_1, \cdots, a_s \quad \mathsf{Z}$, the equation*

$$a_1 x_1 + \cdot + a_s x_s = 0 \tag{6.8}$$

is partition regular over N if and only if there exists $\quad = F \quad [1, S]$ for which $\sum_{i \quad F} a_i = 0$.

Rado also characterized which finite, not necessarily homogeneous, linear systems of equations are partition regular.

Theorem 6.2.5 (Rado, [Rad33]). *Given $\boldsymbol{M} \in M_{m\times n}(\mathbb{Z})$ and $(b_1, \cdots, b_n) \in \mathbb{Z}^n$, the equation*

$$\boldsymbol{M}(x_1, \cdots, x_n)^T = (b_1, \cdots, b_n)^T \tag{6.9}$$

is partition regular over $\mathbb{Z}$ if and only if equation (6.9) admits an integral solution in which $x_1 = x_2 = \cdots = x_n$.

We point out to the reader that in Theorem 6.2.5 it is possible to only obtain partition regularity in a trivial sense. For example, since $(x, y) = (0, 0)$ is the only solution the system of equations

$$\begin{array}{rcrcl} x & - & y & = & 0 \\ x & - & 2y & = & 0 \end{array}, \tag{6.10}$$

we see that the system is not partition regular over $\mathbb{Z} \setminus \{0\}$ even though it is partition regular over $\mathbb{Z}$. As we will see in Section 6.5, Corollary 6.2.4 and Theorem 6.2.5 can be used in conjunction to determine whether or not a single linear equations is partition regular over $\mathbb{N}$ or $\mathbb{Z}$ in a nontrivial fashion.

Theorem 6.2.6 (Bergelson, [Ber96], page 53). *If $p(x) \in \mathbb{Z}[x]$ satisfies $p(0) = 0$, then the equation $x - y = p(z)$ is partition regular over $\mathbb{N}$.*

The theory of *ultrafilters* has been very useful in the study of Ramsey theory and partition regular equations. We briefly recall some basic facts here and give a more detailed introduction in Section 6.9.

Definition 6.2.7. *Given a set S let $\mathcal{P}(S)$ be the power set of the S. $p \subseteq \mathcal{P}(S)$ is an* ***ultrafilter*** *over S if it satisfies the following properties:*

(i) $\emptyset \notin p$.

(ii) If $A \in p$ and $A \subseteq B$ then $B \in p$.

(iii) If $A, B \in p$ then $A \cap B \in p$.

(iv) For any $A \subseteq \mathbb{N}$, either $A \in p$ or $A^c \in p$.

βS denotes the space of all ultrafilters over S.

It is often useful to think about βS as the set of finitely additive $\{0, 1\}$-valued measures on the set S. For now, we only require the following facts about ultrafilters. First, we see that for any finite partition of $S = \bigcup_{i=1}^{r} C_i$ and any ultrafilter p, there exists $1 \leq i_0 \leq r$

for which $C_i \in p$ if and only if $i = i_0$. In fact, for any $A \in p$ and any finite partition $A = \bigcup_{i=1}^{r} C_i$, there exists $1 \leq i_0 \leq r$ for which $C_i \in p$ if and only if $i = i_0$. Secondly, we note that if p is an ultrafilter, $A \in p$ and $B \notin p$, then $A \setminus B = A \cap B^c \in p$. Lastly, we require the existence of a special kind of ultrafilter.

Theorem 6.2.8 (cf. Theorem 6.9.12)**.** *There exists an ultrafilter $p \in \beta\mathbb{N}$ with the following properties:*

(i) For any $A \in p$ and $\ell \in \mathbb{N}$, there exists $b, g \in A$ with $\{bg^j\}_{j=0}^{\ell} \subseteq A$.

(ii) For any $A \in p$ and $h, \ell \in \mathbb{N}$, there exists $a, d \in \mathbb{N}$ for which $\{hd\} \cup \{ha + id\}_{i=-\ell}^{\ell} \subseteq A$.

(iii) For any $s \in \mathbb{N}$, we have $s\mathbb{N} \in p$.

The proof of Theorem 6.2.8 requires more technical knowledge about ultrafilters. Since this technical knowledge is not needed in Sections 6.2-6.8, we defer the proof of Theorem 6.2.8 to Section 6.9.

6.3 On the Partition Regularity of $ax + by = cwz^n$ over $\mathbb{N}$ and $\mathbb{Z} \setminus \{0\}$

The purpose of this section is to prove items (i)(b), (ii), (iii)(a), and (iii)(b) of Theorem 6.1.1. We begin by proving item (i)(b) of Theorem 6.1.1 since it is an easy consequence of the knowledge from the existing literature.

Lemma 6.3.1. *If $a, s \in \mathbb{N}$ and $p(x) \in \mathbb{Z}[x]$ satisfies $p(0) = 0$, then the equation*

$$ax - ay = p(z) \tag{6.11}$$

is partition regular over $s\mathbb{N}$.

Proof. Given a partition $s\mathbb{N} = \bigcup_{i=1}^{r} C_i$, we let $\mathbb{N} = \bigcup_{i=1}^{r}(C_i \cap as\mathbb{N})/as$ be a partition of $\mathbb{N}$. By Theorem 6.2.6, we see that there exists $1 \leq i_0 \leq r$ and $x, y, z \in (C_{i_0} \cap as\mathbb{N})/as$ for which

$$x - y = ca^{n-1}s^n z^{n+1}. \tag{6.12}$$

The desired result in this case follows from the fact that $asx, asy, asz \in C_{i_0}$ and

$$a(asx) - a(asy) = c(asz)^{n+1}. \tag{6.13}$$

□

Corollary 6.3.2 (Item (i)(b) of Theorem 6.1.1)**.** *For any $a, c \in \mathbb{Z} \setminus \{0\}$ and $m, n, s \in \mathbb{N}$ the equation*

$$ax - ay = cw^m z^n \tag{6.14}$$

is partition regular over $s\mathbb{N}$.

Proof. A consequence of Lemma 6.3.1 is that the equation $ax - ay = cz^{m+n}$ is partition regular over $s\mathbb{N}$, so the desired result follows from taking $w = z$. □

We now provide a simple lemma that will not be used later on in the paper, but helps provide some context for items (ii) and (iii) of Theorem 6.1.1.

Lemma 6.3.3. *Given $a, c \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$, the equation*

$$ax = cwz^n \tag{6.15}$$

is partition regular over $\mathbb{Q} \setminus \{0\}$ if and only if $\frac{a}{c}$ is an nth power in $\mathbb{Q}$.

A short proof of Lemma 6.3.3 can be obtained through the use of Theorem 3 of [LR20] by viewing $\mathbb{Q} \setminus \{0\}$ as a $\mathbb{Z}$-module. We choose to give a slightly longer proof since it familiarizes the reader with techniques that will be used repeatedly throughout the rest of the paper. We recall that for a prime $p \in \mathbb{N}$, $v_p : \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$ is the p-adic valuation and is given by $v_p(\frac{t}{s}) = k$ such that p does not divide the numerator or denominator of $\frac{t}{s}p^{-k}$ after simplification.

Proof of Lemma 6.3.3. For the first direction, let us assume that $\frac{a}{c}$ is not an nth power in $\mathbb{Q}$, and let p be a prime for which $n \nmid v_p(\frac{a}{c})$. Let $\mathbb{Q} \setminus \{0\} = \bigcup_{i=1}^{n} C_i$ be the partition given by

$$C_i = \{\frac{t}{s} \in \mathbb{Q} \setminus \{0\} \mid v_p(\frac{t}{s}) \equiv i \pmod{n}\}. \tag{6.16}$$

We see that if $w, x, z \in C_{i_0}$ for some $1 \leq i_0 \leq n$, then

$$v_p(ax) - v_p(cwz^n) \equiv v_p(\frac{a}{c}) + v_p(x) - v_p(w) - nv_p(z) \equiv v_p(\frac{a}{c}) \not\equiv 0 \pmod{n}, \tag{6.17}$$

so we cannot have $ax = cwz^n$.

For the next direction, let us assume that $\frac{a}{c} = (\frac{u}{v})^n$ for some coprime $u, v \in \mathbb{Z} \setminus \{0\}$. Let p be an ultrafilter satisfying the conditions of Theorem 6.2.8 and let $\mathbb{Z} \setminus \{0\} = \bigcup_{i=1}^{r} C_i$

be a partition. By condition (i) of Theorem 6.2.8, we see that for every $A \quad p$, there exists $x, w, z \quad A$ for which $x = wz^n$. We observe that

$$v\mathsf{Z} \setminus \{0\} = \bigcup_{i=1}^{r} \frac{v}{u}(C_i \cap u\mathsf{Z} \setminus \{0\}) \tag{6.18}$$

is a partition, so we may assume without loss of generality that $\frac{v}{u}(C_1 \cap u\mathsf{Z} \setminus \{0\}) \quad p$. It follows that there exists $x, w, z \quad C_1 \cap u\mathsf{Z} \setminus \{0\}$ for which

$$(\frac{v}{u}x) = (\frac{v}{u}w)(\frac{v}{u}z)^n \qquad x = (\frac{v}{u})^n wz^n = \frac{c}{a}wz^n \qquad ax = cwz^n. \tag{6.19}$$

The desired result folows after recalling that $\mathsf{Z} \setminus \{0\} \quad \mathsf{Q} \setminus \{0\}$. □

Our next result, Lemma 6.3.4, is the basis for proving item (ii) of Theorem 6.1.1. While Lemma 6.3.4 is an immediate corollary of Theorem 2.11 of [DNLB18], we decide to give an independent proof for the sake of completeness and to further familiarize the reader with methods that will be used later on in this paper.

Lemma 6.3.4. *Let $p \quad \beta\mathsf{N}$ be an ultrafilter satisfying the conditions of Theorem 6.2.8. For any $A \quad p$, $a, b \quad \mathsf{Z} \setminus \{0\}$ and $n \quad \mathsf{N}$, the equation*

$$ax + by = cwz^n \tag{6.20}$$

has a solution in A if $c \quad \{a, b, a+b\}$.

Proof. Let

$$A \; = \{v \quad A \mid v = wz^n \text{ for some } z, w \quad A\}. \tag{6.21}$$

Since $A \quad p$, to see that $A \; = A \setminus (A \setminus A \;) \quad p$ it suffices to observe that $A \setminus A \; / \, p$ because $A \setminus A$ does not satisfy condition (i) of Theorem 6.2.8. Our first case is when $c = a + b$, and in this case we let $x \quad A$ be arbitrary and let $w, z \quad A$ be such that $x = wz^n$. Since

$$ax + bx = cx = cwz^n, \tag{6.22}$$

we see that x, x, w, z is a solution to equation (6.20) coming from A. For our second case it suffices to consider $c = a$ since the case of $c = b$ is handled similarly. By replacing a, b, c with $-a$, $-b$, $-c$ if necessary, we may assume without loss of generality that $a > 0$. Observe that $A_a := A \; \cap a\mathsf{N} \quad p$ since $A \;, a\mathsf{N} \quad p$ and consider

$$A = \{x_1 \in A_a \mid \text{there exists } x_2 \in A_a \text{ satisfying } x_1 + \lambda\frac{x_2}{a} \in A_a \; \forall \lambda \in [-|b|, |b|]\}. \quad (6.23)$$

Since $A_a \in p$, to see that $A = A_a \setminus (A_a \setminus A) \in p$ it suffices to observe that $A_a \setminus A \notin p$ because $A_a \setminus A$ does not satisfy condition (ii) of Theorem 6.2.8 with $(h, \ell) = (c, b)$. Now let $x_1 \in A$ be arbitrary and let $x_2 \in A_a$ be as in equation (6.23). then we observe that

$$ax_1 + bx_2 = a(x_1 + b\frac{x_2}{a}). \tag{6.24}$$

Since $x_1 + b\frac{x_2}{a} \in A$, so we may pick $w, z \in A$ for which $x_1 + b\frac{x_2}{a} = wz^n$. In this case we observe that

$$ax_1 + bx_2 = c(x_1 + b\frac{x_2}{a}) = cwz^n, \tag{6.25}$$

so x_1, x_2, w, z is a solution to equation (6.20) coming from A. □

Theorem 6.3.5 (Item (ii) of Theorem 6.1.1)**.** *If $a, b, c \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$ are such that one of $\frac{a}{c}, \frac{b}{c}$, or $\frac{a+b}{c}$ is an nth power in $\mathbb{Q}_{\geq 0}$, then the equation*

$$ax + by = cwz^n \tag{6.26}$$

is partition regular over $s\mathbb{N}$ for any $s \in \mathbb{N}$. If one of $\frac{a}{c}, \frac{b}{c}$, or $\frac{a+b}{c}$ is an nth power in $\mathbb{Q}$, then equation (6.26) is partition regular over $s\mathbb{Z} \setminus \{0\}$ for any $s \in \mathbb{N}$.

Proof. Let $d \in \{a, b, a+b\}$ be such that $\frac{d}{c} = (\frac{u}{v})^n$ with $u, v \in \mathbb{Z}$. We see that if $d = 0$, then $a = -b$ and the desired result follows from Lemma 6.3.1 after taking $w = z$. Let us now assume that $d = 0$, so we also have that $u = 0$. Using Lemma 6.3.4 we see that if $p \in \beta\mathbb{N}$ is an ultrafilter satisfying the properties of Theorem 6.2.8, then for any $A \in p$ there exists $w, x, y, z \in A$ for which

$$ax + by = dwz^n. \tag{6.27}$$

We now consider the cases of $\frac{u}{v} \in \mathbb{Q}^+$ and $\frac{u}{v} \in \mathbb{Q}$ separately. If $\frac{u}{v} \in \mathbb{Q}^+$ and $s\mathbb{N} = \bigcup_{i=1}^r C_i$ is a partition, then

$$vs\mathbb{N} = \bigcup_{i=1}^r \frac{v}{u}(C_i \cap us\mathbb{N}) \tag{6.28}$$

is also a partition. Similarly, if $\frac{u}{v} \in \mathbb{Q}$ and $s\mathbb{Z} \setminus \{0\} = \bigcup_{i=1}^r C_i$ is a partition, then

$$vs\mathsf{N} = \bigcup_{i=1}^{r}\left(\frac{v}{u}(C_i \cap us\mathsf{N}) \cap vs\mathsf{N}\right) \tag{6.29}$$

is also a partition. In either case, since $vs\mathsf{N} \quad p$, there exists $1 \leq i_0 \leq r$ for which $\frac{v}{u}(C_{i_0} \cap us\mathsf{N}) \cap vs\mathsf{N} \quad p$, so there exist $w, x, y, z \quad C_{i_0} \cap us\mathsf{N}$ for which

$$a(\frac{v}{u}x) + b(\frac{v}{u}y) = d(\frac{v}{u}w)(\frac{v}{u}z)^n \qquad ax + by = d(\frac{v}{u})^n wz^n = cwz^n. \tag{6.30}$$

□

A particularly aesthetic result arises when we set $n = 1$ in Theorem 6.3.5.

Corollary 6.3.6. *For any $a, b, c \quad \mathsf{Z} \setminus \{0\}$ the equation*

$$ax + by = cwz \tag{6.31}$$

is partition regular over $\mathsf{Z} \setminus \{0\}$.

Remark 6.3.7. It is interesting to note that the equation $x + y = -wz$ is partition regular over $\mathsf{Z} \setminus \{0\}$ as a consequence of Theorem 6.3.5, but not over N due to sign obstructions. We are currently unable to determine whether equations such as $2x - 8y = wz^3$ are partition regular over N since there are no sign obstructions preventing the partition regularity.

Now that we have proven (ii) of Theorem 6.1.1, we are ready to state Theorem 6.3.8, which will be a crucial tool in our efforts to prove item (iii) of Theorem 6.1.1. The techniques that we use to prove Theorem 6.3.8 are similar to techniques used in [BLM21], [DNLB18], and [Rad33], to show that certain equations are not partition regular over N. We note that if $p \quad \mathsf{N}$ is a prime and $r, s \quad \mathsf{Z}$ are such that $p \nmid s$, then we define $\frac{r}{s} \equiv rs^{-1} \pmod{p}$.

Theorem 6.3.8. *Given $a, b, c \quad \mathsf{Z} \setminus \{0\}$ and $n \quad \mathsf{N}$, the equation*

$$ax + by = cwz^n \tag{6.32}$$

is not partition regular over $\mathsf{Q} \setminus \{0\}$ if there exists a prime $p > \max(|a| + |b|, |c|)$ for which $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a+b}{c}$ are not nth powers mod p.

Proof. Let $p > \max(|a| + |b|, |c|)$ be a prime for which a, b, and $a + b$ are not nth powers modulo p. Let $\chi : \mathsf{Q} \setminus \{0\} \to [1, p-1]$ be given by

$$\frac{x}{p^{v_p(x)}} \equiv \chi(x) \pmod{p}. \tag{6.33}$$

Note that for all $r, s \in \mathbb{Z}$ we have $\chi(rs) \equiv \chi(r)\chi(s) \pmod p$ and for all nonzero $-p < r < p$ we have $r \equiv \chi(r) \pmod p$. We also see that for all $r, s \in \mathbb{Z}$ we have

$$\chi(r+s) \equiv \begin{cases} \chi(r)+\chi(s) \pmod p & \text{if } v_p(r) = v_p(s) \text{ and } r+s \equiv 0 \pmod p \\ \chi(s) \pmod p & \text{if } v_p(r) > v_p(s) \\ \chi(r) \pmod p & \text{if } v_p(s) > v_p(r) \end{cases}. \quad (6.34)$$

Let $\mathbb{Q} \setminus \{0\} = \bigcup_{i=1}^{p-1} C_i$ be the partition given by $C_i = \chi^{-1}(\{i\})$. Let us assume for the sake of contradiction that there exists $d \in [1, p-1]$ and $w, x, y, z \in C_d$ satisfying equation (6.32). We now have 3 cases to consider. If $v_p(x) = v_p(y)$, then we see that

$$0 \equiv (a+b)d \equiv \chi(a)\chi(x) + \chi(b)\chi(y) \equiv \chi(ax+by) \equiv \chi(cwz^n) \equiv cd^{n+1} \pmod p \quad (6.35)$$

$$\Rightarrow \quad (a+b)c^{-1} \equiv d^n \pmod p, \quad (6.36)$$

which yields the desired contradiction in this case. For our next case we assume that $v_p(x) < v_p(y)$ and note that

$$0 \equiv ad \equiv \chi(a)\chi(x) \equiv \chi(ax+by) \equiv \chi(cwz^n) \equiv cd^{n+1} \pmod p \quad (6.37)$$

$$\Rightarrow \quad ac^{-1} \equiv d^n \pmod p, \quad (6.38)$$

which once again yields a contradiction. Similarly, in our final case when $v_p(x) > v_p(y)$ we have

$$0 \equiv bd \equiv \chi(b)\chi(y) \equiv \chi(ax+by) \equiv \chi(cwz^n) \equiv cd^{n+1} \pmod p \quad (6.39)$$

$$\Rightarrow \quad bc^{-1} \equiv d^n \pmod p, \quad (6.40)$$

which once more yields a contradiction. □

Lemma 6.3.9 (cf. Corollary 6.4.2)**.** *Let n be odd and suppose that $\alpha, \beta, \gamma \in \mathbb{Q}$ are not nth powers, or let n be even and suppose that $\alpha, \beta, \gamma \in \mathbb{Q}$ are not $\frac{n}{2}$th powers with at least one not $\frac{n}{4}$th power if $4 \mid n$. There exist infinitely many primes $p \in \mathbb{N}$ for which α, β, and γ are simultaneously not nth powers modulo p.*

Corollary 6.3.10 (cf. Theorem 6.1.1(iii)(a-b))**.** *Let $n \quad \mathsf{N}$ and $a, b, c \quad \mathsf{Z} \setminus \{0\}$ be such that either n is odd and none of $\frac{a}{c}, \frac{b}{c}, \frac{a+b}{c}$ are nth powers in Q, or $n = 4, 8$ is even and none of $\frac{a}{c}, \frac{b}{c}$, or $\frac{a+b}{c}$ are $\frac{n}{2}$th powers in Q. Then the equation*

$$ax + by = wz^n \tag{6.41}$$

is not partition regular over $\mathsf{Q} \setminus \{0\}$.

Proof. By Theorem 6.3.8 it suffices to construct a prime $p > \max(|a| + |b|, |c|)$ for which none of $\frac{a}{c}, \frac{b}{c}$, or $\frac{a+b}{c}$ are perfect nth powers modulo p. Firstly, we see that if n is odd, then we may use Lemma 6.3.9 to show that the desired prime p exists. Next, we see that if $n = 2m$ with m odd, then none of $\frac{a}{c}, \frac{b}{c}, \frac{a+b}{c}$ are mth powers by assumption, so we may once again use Lemma 6.3.9 to find the a prime $p > \max(|a| + |b|, |c|)$ for which none of $\frac{a}{c}, \frac{b}{c}$, or $\frac{a+b}{c}$ are mth powers mod p. We note that none of $\frac{a}{c}, \frac{b}{c}, \frac{a+b}{c}$ are nth powers mod p since $m|n$, so p is the desired prime in this case. Lastly, we see that if $4|n$ and $n \geq 12$, then $\frac{n}{4} \geq 3$, so by Fermat's Last Theorem, at least one of $\frac{a}{c}, \frac{b}{c}$, or $\frac{a+b}{c}$ is not an $\frac{n}{4}$th power in Q, so we may once again use Lemma 6.3.9 to show that the desired prime p exists. □

6.4 A Variant of the Grunwald-Wang Theorem

In this section we assume that the reader has had an introduction to algebraic number theory. Specifically, we assume familiarity with the content appearing in chapters I, II, and IV of [Lan94] and the Chebotarev Density Theorem. The main goal of this section is to prove Lemma 6.3.9 as Corollary 6.4.2. The reader willing to take the existence of such primes on faith can safely skip this section and the algebraic number theory content appearing here. We first handle the odd exponent case as some aspects of the argument are simplified and very general. Afterwards, we add a few details to handle the even exponent case.

We briefly recall some of the concepts we will need. We call K a number field if it is a finite field extension of Q. We write O_K for the ring of integers of K, which is the integral closure of Z in K. This is a Dedekind domain, so nonzero ideals factor uniquely into a product of prime ideals.

Given an extension of number fields L/K, one can ask how a prime ideal $\mathfrak{p} \quad O_K$ factors in O_L. We have $\mathfrak{p}O_L = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_g^{e_g}$ for some prime ideals $\mathfrak{q}_i \quad O_L$. Recall that $e(\mathfrak{q}_i/\mathfrak{p}) := e_i$ is the ramification degree of $\mathfrak{q}_i$ over $\mathfrak{p}$ and $f_i = f(\mathfrak{q}_i/\mathfrak{p}) := [O_L/\mathfrak{q}_i : O_K/\mathfrak{p}]$ is the inertia degree of $\mathfrak{q}_i$ over $\mathfrak{p}$. We say $\mathfrak{p}$ is unramified in O_L, or just in L, if $e_i = 1$ for all i. It is a fact that only finitely many prime ideals of O_K are ramified in L.

These invariants are bounded via the following classic formula:

$$[L : K] = \sum_{i=1}^{g} e_i f_i. \tag{6.42}$$

In the case L/K is Galois, we have all of the e_i's and f_i's are equal, so in fact e_i and f_i both divide $[L : K]$.

To compute these numbers in practice, one uses modular arithmetic and factoring polynomials. The process can be summarized as follows. Suppose $L = K(\alpha)$ with $\alpha \quad \mathcal{O}_L$. Then $\mathcal{O}_K[\alpha] \quad \mathcal{O}_L$ and both are finite free $\mathcal{O}_K$-modules of rank $[L : K]$, so $[\mathcal{O}_L : \mathcal{O}_K[\alpha]]$ is finite. If the residue characteristic of $\mathfrak{p} \quad \mathcal{O}_K$ does not divide $[\mathcal{O}_L : \mathcal{O}_K[\alpha]]$, then the factorization behavior of $\mathfrak{p}$ in $\mathcal{O}_L$ can be detected by factoring the minimal polynomial of α mod $\mathfrak{p}$.

More precisely, let $f(x) \quad \mathcal{O}_K[x]$ be the minimal polynomial for α. Then, under the divisibility assumption above, we have

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_g^{e_g} \qquad f(x) \equiv q_1(x)^{e_1} \cdots q_g(x)^{e_g} \bmod \mathfrak{p} \tag{6.43}$$

and $\deg(q_i(x)) = f(\mathfrak{q}_i/\mathfrak{p})$. Hence for all but finitely many $\mathfrak{p} \quad \mathcal{O}_K$, its factoring behavior in $\mathcal{O}_L$ is detected by factoring $f(x)$ mod $\mathfrak{p}$.

When L/K is Galois with group G, there is an important relationship between the arithmetic and algebra of the fields expressed via Frobenius elements. Suppose $\mathfrak{q} \quad \mathcal{O}_L$ divides $\mathfrak{p} \quad \mathcal{O}_K$. Then there exists a unique $\mathrm{Frob}_{\mathfrak{q}/\mathfrak{p}} \quad G$ defined by the property

$$\mathrm{Frob}_{\mathfrak{q}/\mathfrak{p}}(x) \equiv x^q \bmod \mathfrak{q} \tag{6.44}$$

where $q = \#\mathcal{O}_L/\mathfrak{p}$. If $\mathfrak{q}$ and $\mathfrak{q}$ are primes dividing $\mathfrak{p}$ in $\mathcal{O}_L$, then $\mathrm{Frob}_{\mathfrak{q}/\mathfrak{p}}$ and $\mathrm{Frob}_{\mathfrak{q}\ /\mathfrak{p}}$ are conjugate. Conversely, for every $\sigma \quad G$ in the conjugacy class of $\mathrm{Frob}_{\mathfrak{q}/\mathfrak{p}}$, there exists $\mathfrak{q}$ dividing $\mathfrak{p}$ so that $\sigma = \mathrm{Frob}_{\mathfrak{q}\ /\mathfrak{p}}$. Thus we can speak of a well-defined Frobenius conjugacy class $\mathrm{Frob}_{\mathfrak{p}} \quad G$. When G is abelian, Frobenius elements associated to primes in K are therefore well-defined.

In order for this to be useful to us, we need a way to construct primes with given Frobenius elements. The major tool for achieving this is the Chebotarev density theorem.

Theorem 6.4.1 (Chebotarev Density Theorem)**.** *Let L/K be a Galois extension of number fields. Let $C \quad G = \mathrm{Gal}(L/K)$ be a fixed conjugacy class. Then the natural density of primes $\mathfrak{p} \quad K$ with $\mathrm{Frob}_{\mathfrak{p}} \quad C$ is given by $\#C/\#G$, i.e.*

$$\lim_{x \to \infty} \frac{\#\{\mathfrak{p} \quad \mathcal{O}_K : \#\mathcal{O}_K/\mathfrak{p} \leq x, \text{and} \mathrm{Frob}_{\mathfrak{p}} \quad C\}}{\#\{\mathfrak{p} \quad \mathcal{O}_K : \#\mathcal{O}_K/\mathfrak{p} \leq x\}} = \frac{\#C}{\#G}. \tag{6.45}$$

Fix $n > 1$. We turn our attention to studying the polynomial $x^n - a$ for $a \quad \mathcal{O}_K$ and its factoring behavior modulo various primes in $\mathcal{O}_K$. Suppose $a \quad \mathcal{O}_K$ is not an nth power. Suppose and $a = \alpha^d$ for $\alpha \quad \mathcal{O}_{K(\zeta_n)}$ with $d \mid n$ maximal. Set $m = n/d$. Consider the diagram of fields:

$$
\begin{array}{c}
L = C(\alpha^{1/m}) \\
\big| \\
C = K(\zeta_n) \\
\big| \\
K
\end{array}
$$

where ζ_n is a primitive nth root of unity, and $\alpha^{1/m}$ is an arbitrary root of the polynomial $x^m - \alpha$. We will show the field L is a well-defined radical extension of C using the following lemma.

Lemma 6.4.1 ([Lan02] Theorem VI.9.1)**.** *Let k be a field and $a \in k$ nonzero. Assume for all primes $p \mid n$, we have $a \notin k^p$ and if $4 \mid n$, then $a \notin -4k^4$. Then $x^n - a$ is irreducible in $k[x]$.*

From here we deduce the following.

Lemma 6.4.2. *The polynomial $x^m - \alpha$ is irreducible over C. In particular, the field L above is a well-defined radical extension of C, and $[L : C] = m$.*

Proof. By maximality of d, we see for all primes $p \mid n$ we have $\alpha \notin C^p$, so the polynomial $x^m - \alpha$ satisfies the first criteria of Lemma 6.4.1, and the polynomial is irreducible as long as $4 \nmid n$. In the case $4 \mid n$, we check $\alpha \notin -4C^4$. Suppose otherwise. Then $\alpha = -4\beta^4$ for some $\beta \in C$. But since $4 \mid n$ and $\zeta_n \in C$, we have $\sqrt{-1} = \zeta_4 \in C$, so $\alpha = (\zeta_4 \cdot 2 \cdot \beta^2)^2$, and $a = (\zeta_4 \cdot 2 \cdot \beta^2)^{2d}$ contradicting maximality of d. In either case, we find $x^m - \alpha$ is irreducible.

For the statement on degrees, we use Kummer theory. Recall this tells us that since C contains all mth roots of unity, extensions of the form $C(\alpha^{1/m})/C$ are cyclic of degree equal to the order of α in $C^\times/C^{\times,m}$. But we have just showed that α is not a dth power for any $d \mid m$, so its order in this group is m. □

Remark 6.4.3. The key idea of our argument is as follows. We will use density arguments to produce a prime ideal $\mathfrak{p}$ in the ring of integers of $C = K(\zeta_n)$ modulo which $x^m - \alpha$ has no root. Given such a $\mathfrak{p}$, the going down theorem provides a prime ideal of $\mathcal{O}_K$ with the same property. In fact, if we can bound the density of such $\mathfrak{p}$ well enough, then repeating with b and c can yield a density bound on the set of prime ideals $\mathfrak{p}$ where at least one of a, b, c are nth powers. If this density is less than 1, then the lemma will be established in that case.

The setup is as follows. Let $a = \alpha^{d_a}$; $b = \beta^{d_b}$; and $c = \gamma^{d_c}$ with the d's maximal. Set $m_a = n/d_a$, etc. Let $\mathfrak{p} \subset \mathcal{O}_K$. Then we have $x^n - a$ has a root mod $\mathfrak{p}$ if and only if $x^m - \alpha$

has a root mod $\mathfrak{p}$, and (with finitely many exceptions for $\mathfrak{p}$) the density of such $\mathfrak{p}$ correspond to the density of $\mathfrak{p}$ splitting in $L_a = C(\alpha^{1/m_a})$. By the Chebotarev density theorem, the latter is given by $1/[L_a : C]$.

Let $\delta_a := 1/[L_a : C] = 1/m_a$ and similarly define δ_b and δ_c. As mentioned, if $\delta_a+\delta_b+\delta_c < 1$, then there must exist a prime (infinitely many, in fact) in C where none of a, b, c are nth powers. Unfortunately this sum can very well be at least 1 or more, so we will devote most of the rest of this section handling those cases.

First, we gather some results to rule out the case that $\delta_a = 1$, at least when $K = \mathsf{Q}$ and many other cases. We recall a fact about the interplay between roots of unity and radical extensions, due to Schinzel. Since Lemmas 6.4.4 and 6.4.5 are generic, we will omit the subscripts and just write m, d, and L in their statements and proofs.

Lemma 6.4.4. *Let ω_m be the number of roots of unity in a field F of characteristic 0. Suppose $x^m - \alpha$ is irreducible over F. Then $F(\alpha^{1/m})/F$ is an abelian Galois extension if and only if $\alpha^{\omega_m} = \beta^m$ for some $\beta \quad F$.*

Proof. See [V́80]. □

This provides us with our first serious condition on δ_a, by restricting the degree $[L : C]$.

Lemma 6.4.5. *Let ω_n be the number of nth roots of unity in K. Suppose $m = [L : C] = 1$ with the notation as above. Then*

$$a^{\omega_n} = k^n \tag{6.46}$$

for some $k \quad K$.

Proof. Because C/K is a cyclotomic extension, it is abelian. But since $m = 1$, we have $a^{1/n} = \alpha \quad C$. Hence $K(a^{1/n})/K$ is abelian. By Lemma 6.4.4, we then have $a^{\omega_n} = k^n$ for some $k \quad K$. □

In particular, we can control the size of m by the assumptions we make on α, and hence a, in K. We will carry the details out towards the end of the proof of the main lemma. For now, we investigate the density of primes where both a and b are nth powers.

Lemma 6.4.6. *Let m_a and m_b be as in Remark 6.4.3. The density of primes in $K(\zeta_n)$ where both a and b are nth powers modulo is given by at least $1/(m_a \cdot m_b)$.*

Proof. Since both $C(\alpha^{1/m_a})/C$ and $C(\beta^{1/m_b})/C$ are Galois extensions, their composite field $C(\alpha^{1/m_a}, \beta^{1/m_b})$ is Galois over C and has degree at most $m_a \cdot m_b$. The primes where both $x^{m_a} - \alpha$ and $x^{m_b} - \beta$ have roots modulo correspond to those with trivial Frobenius element in $G = \mathrm{Gal}(C(\alpha^{1/m_a}, \beta^{1/m_b})/C)$. By Chebotarev, these have density $1/\#G \geq 1/(m_a \cdot m_b)$. □

We can now prove the main existence result.

Lemma 6.4.7. *Let K be a number field and ω_n the number of nth roots of unity in K. Let $a, b, c \quad O_K$.*

(i) Suppose n is odd, and that a^{ω_n} is not an nth power in O_K, and similarly for b, c; or

(ii) Suppose n is even, and a, b, c satisfy the same conditions as in (i), but $a^{2\omega_n}$ is also not an nth power.

Then there exists infinitely many primes of K modulo which none of a, b, c are nth powers.

Proof. Recall that $C = K(\zeta_n)$ and $L_a = C(\alpha^{1/m_a})$. Let $\delta_{a,b}$ be the density of primes of C modulo which both a and b are nth powers. By Lemma 6.4.6, we have $\delta_{a,b} \geq 1/(m_a \cdot m_b)$. Similarly we write $\delta_{a,b,c}$ for the density of primes modulo which all three are nth powers. Also let Δ be the density of primes modulo which at least one of a, b, c is an nth power. We want to use inclusion-exclusion to bound Δ. We have

$$\Delta = \delta_a + \delta_b + \delta_c - \delta_{a,b} - \delta_{b,c} - \delta_{a,c} + \delta_{a,b,c}.$$

Suppose without loss of generality that $\delta_{a,c}$ is minimal among the densities for the possible pairs. Then $\delta_{a,b,c} \leq \delta_{a,c}$, so we have

$$\Delta \leq \delta_a + \delta_b + \delta_c - \delta_{a,b} - \delta_{b,c}, \tag{6.47}$$

$$\leq 1/m_a + 1/m_b + 1/m_c - 1/(m_a \cdot m_b) - 1/(m_b \cdot m_c), \tag{6.48}$$

using the bound from Lemma 6.4.6. We now split into cases based on the parity of n.

(i) We handle the case when n is odd first. We see by our assumptions and Lemma 6.4.5, we have $m_a, m_b, m_c > 1$. Since $m_a = [L_a : C]$ must divide n and n is odd, we have $m_a, m_b, m_c \geq 3$. So then $\Delta \leq 1/3 + 1/3 + 1/3 - 1/9 - 1/9 = 7/9 < 1$.

(ii) When n is even, there are few more exceptional cases for $(\delta_a, \delta_b, \delta_c)$ which we handle now:

- $(1/k,\ 1/3,\ 1/2)$, $3 \leq k \leq 6$: From our bound above, we get

$$\Delta \leq \frac{1}{k} + \frac{1}{3} + \frac{1}{2} - \frac{1}{6} - \frac{1}{2k} = \frac{2}{3} + \frac{1}{2k} < 1. \tag{6.49}$$

- $(1/k,\ 1/2,\ 1/2)$, $k \geq 3$: From our bound above, we get

$$\Delta \leq \frac{1}{k} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} - \frac{1}{2k} = 3/4 + \frac{1}{2k} < 1. \tag{6.50}$$

- $(1/2, 1/2, 1/2)$: This case is genuine cause for concern. For example, it is possible to have 3 quadratic extensions of a field where every prime in the base splits in at least one of them (consider $\mathbb{Q}(\sqrt{a}), \mathbb{Q}(\sqrt{b})$, and $\mathbb{Q}(\sqrt{ab})$). However, we rule this out by the additional assumption on a. Suppose $m_a = 2$ so that $\delta_a = 1/2$. Then since $x^n - a = (x^2 - \alpha)f(x)$ for some $f(x) \in C[x]$, we are justified in writing $\alpha = a^{2/n}$. We have $K((a^2)^{1/n})/K$ is abelian since $K((a^2)^{1/n}) = K(\alpha) \subseteq C$, and C/K is abelian as it is a cyclotomic extension. Applying Lemma 6.4.4, we see that

$$a^{2\omega_n} = k^n \tag{6.51}$$

for some $k \in K$, contradicting the original assumption on a.
- Otherwise, we have already $\delta_a + \delta_b + \delta_c < 1$, so we get $\Delta < 1$.

In all cases we obtain that the density of primes p where at least one of a, b, c is an nth power modulo p is less than 1, so there exists a set of primes of positive density where none are nth powers. □

In the case $K = \mathbb{Q}$, this specializes to a more pleasant form. We examine an equally pleasant generalization for $\alpha, \beta, \gamma \in \mathbb{Q}$ rather than $a, b, c \in \mathbb{Z}$ as demanded by the applications from Section 6.3. We remind the reader that if $p \in \mathbb{N}$ is a prime and $r, s \in \mathbb{Z}$ are such that $p \nmid s$, then $\frac{r}{s} \equiv rs^{-1} \pmod{p}$.

Corollary 6.4.2. *Let $\alpha, \beta, \gamma \in \mathbb{Q}$.*

(i) Suppose n is odd and α, β, γ are not nth powers; or

(ii) Suppose n is even, α, β, γ are not $\frac{n}{2}$th powers, and α is not an $\frac{n}{4}$th power if $4 \mid n$.

Then there exists infinitely many primes $p \in \mathbb{N}$ modulo which none of α, β, γ are nth powers.

Proof. Let $c \in \mathbb{N}$ be such that for $\alpha' := c^n\alpha, \beta' := c^n\beta$, and $\gamma' := c^n\gamma$ we have $\alpha', \beta', \gamma' \in \mathbb{Z}$. Since $\omega_n = 1$ when n is odd and $\omega_n = 2$ when n is even we may apply Lemma 6.4.7 to find infinitely many primes $p \in \mathbb{N}$ for which α', β', and γ' are noth nth powers modulo p. If $p \in \mathbb{N}$ is a prime for which $p \nmid c$, then α' is an nth power modulo p if and only if α is an nth power modulo p, and similarly for β' and γ'. It follows that there are infinitely many primes p for which α, β, and γ are also not nth powers modulo p. □

Remark 6.4.3. *When working over $\mathbb{Q}$, as long as $n = 4, 8$, when specialized to our situation where $c = a+b$, the condition that one of the three numbers not be an $\frac{n}{4}$th power is automatic by Fermat's Last Theorem.*

Now we give a few more remarks about the subtleties that arose in this proof, and what obstacles prevent pushing it further, both over $\mathbb{Q}$ and for general number fields K.

(i) *It could happen that $x^n - a$ has a root mod all primes $\mathfrak{p}$ O_K even though a is not an nth power. For example, this happens when $K =$ Q; the polynomial $x^8 - 16$ has a root mod p for all primes p Z, but 16 is not an 8th power. The Grunwald-Wang theorem says that if $x^n - a$ has a root mod p for all primes p, then n must be even of a special form, determined by K. This is ruled out above since 16 is a perfect $8/2 = 4$th power.*

(ii) *Even if one were to exclude the situation arising in the Grunwald-Wang theorem, it is possible that $(x^n - a)(x^n - b)$ could have a root mod p for all primes p, despite neither factor having this behavior. We consider 3 such examples when $K =$ Q, each of which also suggests the necessity of the conditions in item (ii) of Lemma 6.4.7 and item (ii) or Corollary 6.4.2.*

   (a) *The polynomial $(x^{12} - 3^6)(x^{12} - \beta^4)$ has this feature for any β N, because $(x^2 + 3)(x^3 - \beta)$ has a root mod p for all p Z. The problem here is that $x^{12} - 3^6$ has a quadratic factor, which lives inside Q(ζ_{12}). In the language above, this means that $e = 1$, so the density argument won't work. (cf. [HLS14])*

   (b) *Since $x^8 - 16 = (x^4 - 4)(x^4 + 4)$ and $x^8 - 16$ has a root modulo p for any prime p N, one of 4 and -4 is a 4th power modulo p.*

   (c) *Since 36 is a fourth power modulo p for any prime $p \equiv 13 \pmod{24}$ and 9 is a fourth power modulo p for any prime $p \equiv 13 \pmod{24}$, we see that $(x^4 - 36)(x^4 - 9)$ has a root modulo p for any prime p N.*

   For general K, one imagines it only gets harder to determine whether $x^n - a$ has a factor whose splitting field lies in a cyclotomic extension. The proof above implies that if K doesn't have many roots of unity, then there are mild conditions on a, b, c to get the existence of the desired prime.

(iii) *When K doesn't have unique factorization, we lose some of the power a orded by Lemma 6.4.5 because we cannot ensure that if $a^x = b^y$, then a is a perfect $y/(x, y)$th power. We can pass to ideals generated by a and b, and use the fact that O_K is Dedekind, so its ideals satisfy unique factorization. But this only says that the ideal generated by a is a perfect $y/(x, y)$th power, and this need not imply a has the same property. For example, if $K =$ Q$(\sqrt{-5})$ and $\mathfrak{p} = (2, 1 + \sqrt{-5})$ O_K, then $\mathfrak{p}^2 = (2)$, but $2 = \alpha^2$ for any α O_K.*

(iv) *When K has lots of units, it can also be di cult to deduce a is a perfect $y/(x, y)$th power given $a^x = b^y$. For example, the ideal generated by (-4) in Z is the square of the ideal (2), but -4 is of course not a perfect square. It's possible one could say more here by trying to control the units of O_K, but this seems a bit intimidating and not of immediate interest.*

We see that Corollary 6.4.2 is not useful when $n = 2$ or $n = 4$, so we will address the case of $n = 2$ separately and observe the implications that it has for the case of $n = 4$. We first require a lemma for constructing primes modulo which certain numbers are not squares.

Lemma 6.4.8. *Suppose that $\alpha, \beta, \gamma \quad \mathbb{Q}$ are not squares, and $\alpha\beta\gamma$ is also not a square. There exists a prime $p \quad \mathbb{N}$ for which α, β, and γ are not squares modulo p.*

Proof. Suppose on the contrary that no such prime exists. Let $\left(\frac{a}{p}\right)$ be the Legendre symbol and let $S_{\alpha,\beta}$ be the set of primes p for which

$$\left(\frac{\alpha}{p}\right) = \left(\frac{\beta}{p}\right) = -1. \tag{6.52}$$

By our assumption, we have $\left(\frac{\gamma}{p}\right) = 1$ for all $p \quad S_{\alpha,\beta}$ since otherwise we would have produced a prime with the desired features. Using similar notation for the other pairs, we see that $S_{\alpha,\beta} \cap S_{\beta,\gamma} =$? because for primes p in the first set, we have $\left(\frac{\alpha}{p}\right) = -1$ but for p in the second set we have $\left(\frac{\alpha}{p}\right) = 1$.

Thus the density of $S := S_{\alpha,\beta} \quad S_{\beta,\gamma} \quad S_{\gamma,\alpha} \geq 3/4$ as the density of each $S_{\alpha,\beta}$ is at least $1/4$ by Quadratic Reciprocity. But for each $p \quad S$, we have

$$\left(\frac{\alpha\beta\gamma}{p}\right) = 1. \tag{6.53}$$

Hence a set of primes of density $> 1/2$ split in the extension $\mathbb{Q}(\sqrt{\alpha\beta\gamma})$. But by Quadratic Reciprocity, the set of primes which split in a degree 2 extension has density $1/2$. Thus we have $[\mathbb{Q}(\sqrt{\alpha\beta\gamma}) : \mathbb{Q}] = 1$, so we must have $\alpha\beta\gamma$ is a square in $\mathbb{Q}$, contradicting the original assumption. □

Corollary 6.4.9 (cf. Theorem 6.1.1(iii)(c)). *If $a, b, c \quad \mathbb{Z} \setminus \{0\}$ are such that $\frac{a}{c}, \frac{b}{c}, \frac{a+b}{c}$, and $(\frac{a}{c})(\frac{b}{c})(\frac{a+b}{c})$ are not squares, then for any $n \quad \mathbb{N}$ the equation*

$$ax + by = cwz^{2n} \tag{6.54}$$

is not partition regular over $\mathbb{Q}$.

Proof. We want to use Theorem 6.3.8, so we produce a prime satisfying the conditions there. In particular, we want a prime p such that $\alpha := \frac{a}{c}, \beta := \frac{b}{c}$, and $\gamma := \frac{a+b}{c}$ are not perfect squares in $\mathbb{Z}/p\mathbb{Z}$. Noting that α, β, γ, and $\alpha\beta\gamma$ are not squares in $\mathbb{Q}$, we see that the existence of our desired prime p is a consequence of Lemma 6.4.8. □

We may also use Lemma 6.4.8 to obtain a strengthening of the special case of Corollary 6.4.2 in which there are 2 variables instead of 3, which will be of use in Section 6.7 when we determine necessary conditions for some systems of equations to be partition regular.

Lemma 6.4.10. *Let α, β Q and n N. Suppose that one of the following conditions holds:*

(i) $4 - n$ *and neither of α, β are nth powers.*

(ii) $4|n$ *and neither of α, β are $\frac{n}{2}$th powers.*

Then there exists infinitely many primes p N modulo which neither α, β are nth powers.

Proof. We begin by proving the desired result for item (i) and observe that the only case not handled by Corollary 6.4.2 is when $n = 2m$ with m odd and at least one of α, β is an mth power. Suppose $\alpha = x^m$. Write $\beta = y^d$ for d maximal. By similar density arguments as before, the only edge case is when $d = m$, otherwise the sum of densities of primes where at least one is an nth power will be strictly less than 1. In the case $d = m$, since m is odd, we see that

$$\left(\frac{\alpha}{p}\right) = \left(\frac{x}{p}\right) \text{ and } \left(\frac{\beta}{p}\right) = \left(\frac{y}{p}\right).$$

Letting $\gamma = \alpha = x$ and $\beta = y$, we see that none of α, β, γ, or $\alpha\beta\gamma = \alpha^2\beta$ are squares in Q, so by Lemma 6.4.8 there exists infinitely many primes p such that x, y are not squares mod p, and thus α, β are not nth powers mod p.

To see that the desired result holds for item (ii) we consider the cases of $n = 4$ and $n = 4$ separately. When $n = 4$, we let γ Q be any element that is not a $\frac{n}{4}$th power and apply Corollary 6.4.2. When $n = 4$, we see that neither of α or β are squares, so by item (i) there exists infinitely many primes p N modulo which neither of α, β are squares. □

We observe that Item (ii)(b) of Remark 6.4.3 tells us that for any prime p and odd number m, one of 4^m and -4^m will be a $4m$th power modulo p, which justifies our assumptions in item (ii) of Lemma 6.4.10.

6.5 Reduction to the Case min(m,n) = 1

The main purpose of this section is to show that the equation $ax + by = cw^m z^n$ is not partition regular over Z \ {0} if a, b, c Z \ {0}, $a + b = 0$, and $n, m \geq 2$. Afterwards, we will show that the equation $ax + by = cwz^n$ is not partition regular over Z \ {0} for some values of a, b, c, and n that are not already addressed by Theorem 6.1.1.

Theorem 6.5.1 (cf. Theorem 6.1.1(i)(a)). *If a, b, c Z \ {0} and n, m N are such that $a + b = 0$ and $n, m \geq 2$, then the equation*

$$ax + by = cz^n w^m \tag{6.55}$$

is not partition regular over Z \ {0}.

In order to prove Theorem 6.5.1 we will use item (1) of Theorem 2.19 of [BLM21]. We now review the definitions necessary to state and use this result.

Definition 6.5.2. *Given a polynomial P $\quad \mathbb{Z}[x_1, \cdots, x_n]$, we let **Supp($P$)** denote the collection of α $\quad \mathbb{N}_0^n$ for which*

$$P(x_1, \cdots, x_n) = \sum_{\alpha \quad \mathrm{Supp}(P)} c_\alpha x^\alpha. \tag{6.56}$$

Example 6.1. If $P(x, y) = x^2 + y^2$, we have $\mathrm{Supp}(P) = \{(2, 0), (0, 2)\}$.

Example 6.2. If $P(x, y, z) = xyz + 1$, we have $\mathrm{Supp}(P) = \{(0, 0, 0), (1, 1, 1)\}$.

Example 6.3. If $P(w, x, y, z) = wx+y+z^2$, we have $\mathrm{Supp}(P) = \{(1, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 2)\}$.

Definition 6.5.3 (cf. Def. 2.6 in [BLM21])**.** *Let $\phi : \mathbb{Z}^n \to \mathbb{Z}$ be a positive linear map, i.e.*

$$\phi : (\alpha_1, \cdots, \alpha_n) \to t_1\alpha_1 + \cdots + t_n\alpha_n$$

with $t_1, \cdots, t_n$ $\quad \mathbb{N}_0$.

- *If c is a finite coloring of $\mathbb{N}$, then we say that ϕ is **c-monochromatic** if $\{t_1, \cdots, t_n\}$ is c-monochromatic;*
- *If P $\quad \mathbb{Z}[x_1, \cdots, x_n]$, and $(M_0, \cdots, M_\ell)$ is the increasing enumeration of ϕ(Supp(P)), then the **partition of Supp(P) determined by ϕ** is the ordered tupe $(J_0, \cdots, J_\ell)$, where $J_i = \{\alpha \quad \mathrm{Supp}(P) : \phi(\alpha) = M_i\}$.*

Definition 6.5.4 (cf. Def. 2.7 in [BLM21])**.** *A **Rado partition** of P $\quad \mathbb{Z}[x_1, \cdots, x_n]$ is an ordered tuple $(J_0, \cdots, J_\ell)$ such that, for every finite coloring c of $\mathbb{N}$, there exist infinitely many c-monochromatic positive linear maps $\phi : \mathbb{Z}^n \to \mathbb{Z}$ such that $(J_0, \cdots, J_\ell)$ is the partition of Supp(P) determined by ϕ.*

Definition 6.5.5 (cf. Def. 2.8 in [BLM21])**.** *A **Rado set** for P $\quad \mathbb{Z}[x_1, \cdots, x_n]$ is a set J $\quad$ Supp(P) such that there exists a Rado partition $(J_0, \cdots, J_\ell)$ for P such that $J = J_i$ for some i $\quad \{0, 1, \cdots, \ell\}$*

Definition 6.5.6 (cf. Def. 2.13 in [BLM21])**.** *An **upper Rado functional** of order m for P $\quad \mathbb{Z}[x_1, \cdots, x_n]$ is a tuple $(J_0, \cdots, J_\ell, d_0, \cdots, d_{m-1})$ for some $\ell \geq m$ and $d_0, \cdots, d_{m-1}$ $\mathbb{N}$ such that, for every finite coloring c of $\mathbb{N}$ and for every r $\quad \mathbb{N}$, there exist infinitely many c-monochromatic positive linear maps $\phi : \mathbb{Z}^n \to \mathbb{Z}$, $(\alpha_1, \cdots, \alpha_n) \to t_1\alpha_1 + \cdots + t_n\alpha_n$ such that $(J_\ell, \cdots, J_0)$ is the partition of Supp(P) determined by ϕ, and if $(M_\ell, \cdots, M_0)$ is the increasing enumeration of ϕ(Supp(P)), then $M_i - M_m = d_i$ for i $\quad \{0, 1, \cdots, m-1\}$, and $M_m - M_i \geq r$ for i $\quad \{m+1, \cdots, \ell\}$.*

Definition 6.5.7 (cf. Def. 2.16 in [BLM21]). *A polynomial* $P(x) = \sum_\alpha c_\alpha x^\alpha \quad \mathsf{Z}[x_1, \cdots, x_n]$ *satisfies the* ***maximal Rado condition*** *if for every* $q \geq 2$ *there exists an upper Rado functional* $(J_0, \cdots, J_\ell, d_0, \cdots, d_{m-1})$ *for* P *such that, setting* $d_m = 0$*, the polynomial*

$$g(w) = \sum_{i=0}^{m} q^{d_i} \sum_{\alpha \quad J_i} c_\alpha w^{|\alpha|} \tag{6.57}$$

has a real root in $[1, q]$.

Theorem 6.5.8 (cf. Theorem 2.19 of [BLM21]). *Fix* $P \quad \mathsf{Z}[x_1, \cdots, x_n]$. *If the equation* $P(x_1, \cdots, x_n) = 0$ *is partition regular over* N*, then* P *satisfies the maximal Rado condition.*

We are now ready to begin proving Theorem 6.5.1.

Lemma 6.5.9. *If* $a, b, c \quad \mathsf{Z} \setminus \{0\}$ *and* $n, m \quad \mathsf{N}$ *are such that* $a + b = 0$ *and* $n, m \geq 2$*, then the equation*

$$ax + by = cw^m z^n \tag{6.58}$$

is not partition regular over N.

Proof. Let us fix $a, b, n, m \quad \mathsf{N}$ with $n, m \geq 2$. We will begin by showing that the polynomial $P(w, x, y, z) = cz^n w^m - ax - by$ does not have any upper Rado functional of order $m \geq 1$. From there it will be relatively simple to verify that P does not satisfy the maximal Rado condition. To this end, let us assume for the sake of contradiction that $(J_0, \cdots, J_\ell, d_0, \cdots, d_{m-1})$ is an upper Rado functional of order m for P. Note that $\text{Supp}(P) = \{(m, 0, 0, n), (0, 1, 0, 0), (0, 0, 1, 0)\} = \{M_0, M_1, M_2\}$. By considering all possible orderings of the set $\{M_0, M_1, M_2\}$ and the definition of d_0, we see that for any finite coloring c of N there exists infinitely many c-monochromatic positive linear maps

$$\phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 + t_4\alpha_4, \tag{6.59}$$

for which at least one of equations (6.60)–(6.65) has a solution:

$$\phi(m, 0, 0, n) - \quad \phi(0, 1, 0, 0) = d_0 \qquad mt_1 - t_2 + \quad nt_4 = d_0, \tag{6.60}$$

$$\phi(0, 1, 0, 0) - \quad \phi(m, 0, 0, n) = d_0 \qquad -mt_1 + t_2 - \quad nt_4 = d_0, \tag{6.61}$$

$$\phi(m, 0, 0, n) - \quad \phi(0, 0, 1, 0) = d_0 \qquad mt_1 - t_3 + nt_4 = d_0, \tag{6.62}$$

$$\phi(0, 0, 1, 0) - \quad \phi(m, 0, 0, n) = d_0 \qquad -mt_1 + t_3 - nt_4 = d_0, \tag{6.63}$$

$$\phi(0, 1, 0, 0) - \quad \phi(0, 0, 1, 0) = d_0 \qquad t_2 - t_3 = d_0, \tag{6.64}$$

$$\phi(0, 0, 1, 0) - \quad \phi(0, 1, 0, 0) = d_0 \qquad t_3 - t_2 = d_0. \tag{6.65}$$

Equivalently, for any finite coloring of $\mathbb{N}$ at least one of equations (6.60)–(6.65) must possess infinitely many monochromatic solutions (t_1, t_2, t_3, t_4). By Theorem 6.2.5 we see that any one of equations (6.60)–(6.65) is partition regular over $\mathbb{Z}$ (and hence over $\mathbb{N}$) if and only if there exists a constant solution $t_1 = t_2 = t_3 = t_4 = t$. It follows that equations (6.64) and (6.65) are not partition regular. Furthermore, we see that one of equations (6.60)–(6.63) possess infinitely many monochromatic solutions (t_1, t_2, t_3, t_4) in any finite coloring of $\mathbb{N}$ if and only if one of equations (6.66)–(6.69) below possess infinitely many monochromatic solutions (t_1, t_2, t_3, t_4) in any finite coloring of $\mathbb{N}$:

$$m(t_1 - \frac{d_0}{m+n-1}) - (t_2 - \frac{d_0}{m+n-1}) + n(t_4 - \frac{d_0}{m+n-1}) = 0, \tag{6.66}$$
$$-m(t_1 - \frac{d_0}{m+n-1}) + (t_2 - \frac{d_0}{m+n-1}) - n(t_4 - \frac{d_0}{m+n-1}) = 0, \tag{6.67}$$
$$m(t_1 - \frac{d_0}{m+n-1}) - (t_3 - \frac{d_0}{m+n-1}) + n(t_4 - \frac{d_0}{m+n-1}) = 0, \tag{6.68}$$
$$-m(t_1 - \frac{d_0}{m+n-1}) + (t_3 - \frac{d_0}{m+n-1}) - n(t_4 - \frac{d_0}{m+n-1}) = 0. \tag{6.69}$$

Since $m, n \geq 2$ we may repeatedly use Corollary 6.2.4 to create a finite partition of $\mathbb{N}$ for which the only monochromatic solution to any of equations (6.66)–(6.69) (considered separately, not as a system) is the solution $t_1 = t_2 = t_3 = t_4 = \frac{d_0}{m+n-1}$. It follows that P does not have any upper Rado functionals of order $m \geq 1$. Noting that an upper Rado functional of order 0 is just a Rado partition, we note that the set of Rado partitions of P is

$$\Big\{\Big(\{(m,0,0,n)\},\{(0,1,0,0)\},\{(0,0,1,0)\}\Big),\Big(\{(m,0,0,n)\},\{(0,0,1,0)\},\{(0,1,0,0)\}\Big),$$
$$\Big(\{(0,1,0,0)\},\{(m,0,0,n)\},\{(0,0,1,0)\}\Big),\Big(\{(0,1,0,0)\},\{(0,0,1,0)\},\{(m,0,0,n)\}\Big),$$
$$\Big(\{(0,0,1,0)\},\{(m,0,0,n)\},\{(0,1,0,0)\}\Big),\Big(\{(0,0,1,0)\},\{(0,1,0,0)\},\{(m,0,0,n)\}\Big),$$
$$\Big(\{(0,1,0,0),(0,0,1,0)\},\{(m,0,0,n)\}\Big),\Big(\{(m,0,0,n)\},\{(0,1,0,0),(0,0,1,0)\}\Big)\Big\}.$$

Finally, to see that P does not satisfy the maximal Rado condition, we see that for $q = 2$ (or any other value of $q \geq 2$) we have

$$g(w) = \sum_{i=0}^{m} q^{d_i} \sum_{\alpha \quad J_i} c_\alpha w^{|\alpha|} = \sum_{i=0}^{0} 2^0 \sum_{\alpha \quad J_i} c_\alpha w^{|\alpha|} = \sum_{\alpha \quad J_i} c_\alpha w^{|\alpha|} \quad \{cw^{n+m}, -aw, -bw, -(a+b)w\}, \tag{6.70}$$

so none of the polynomials that $g(w)$ could be contains a root in $[1, 2]$. □

Proof of Theorem 6.5.1. By Lemma 6.5.9 we see that

$$ax + by = cw^m z^n \tag{6.71}$$

and

$$ax + by = (-1)^{n+m-1} cw^m z^n \tag{6.72}$$

are not partition regular over N. Let $\mathsf{N} = \bigcup_{i=1}^{r_1} C_i$ be a partition for which no cell contains a solution to equation (6.71) and let $\mathsf{N} = \bigcup_{i=1}^{r_2} D_i$ be a partition for which no cell contains a solution to equation (6.72). It now suffices to show that no cell of the partition

$$\mathsf{Z} \setminus \{0\} = \left(\bigcup_{i=1}^{r_2} (-D_i) \right) \quad \bigcup_{i=1}^{r_1} C_i \tag{6.73}$$

contains a solution to equation (6.71). It follows from the definition of the C_i that none of them contain a solution to equation (6.71), so let us assume for the sake of contradiction that for some $1 \leq i \leq r_2$ there exist $w\,, x\,, y\,, z \quad -D_i$ which satisfy equation (6.71). Letting $w = -w\,, x = -x\,, y = -y\,$, and $z = -z\,$, we see that $w, x, y, z \quad D_i$ and

$$a(-x) + b(-y) = ax\ + by\ = c(w\ )^m (z\ )^n = c(-w)^m(-z)^n = (-1)^{m+n} cw^m z^n \tag{6.74}$$

$$ax + by = (-1)^{m+n-1} cw^m z^n, \tag{6.75}$$

which yields the desired contradiction. □

We now describe another condition for a polynomial to be partition regular of a particular flavor, involving lower Rado functionals.

Definition 6.5.10 (cf. Def. 2.12 in [BLM21])**.** *A **lower Rado functional** of order $m \quad \mathsf{N} \quad \{0\}$ for $P \quad \mathsf{Z}[x_1, \cdots, x_n]$ is a tuple $(J_0, \cdots, J_\ell, d_1, \cdots, d_m)$ for some $\ell \geq m$ and $d_1, \cdots, d_m \quad \mathsf{N}$ such that, for every finite coloring c of N and for every $N \quad \mathsf{N}$, there exist infinitely many c-monochromatic positive linear maps*

$$\phi : \mathsf{Z}^n \to \mathsf{Z},$$
$$(\alpha_1, \cdots, \alpha_n) \to (t_1\alpha_1 + t_2\alpha_2 + \cdots + t_n\alpha_n),$$

such that $(J_0, \cdots, J_\ell)$ is the partition of Supp(P) determined by ϕ and, if $(M_0, \cdots, M_\ell)$ is the increasing enumeration of $\phi($Supp$(P))$, then $M_i - M_m = d_i$ for $i \quad \{1, 2, \cdots, m\}$, and $M_{m+1} - M_m \geq N$.

Definition 6.5.11 (cf. Def. 2.2 in [BLM21]). *For* $P \quad \mathsf{Z}[x_1, x_2, \cdots, x_n]$ *and* $q \quad \mathsf{N}$*, the equation*

$$P(x_1, x_2, \cdots, x_n) = 0$$

is q-***partition regular*** *if for any* $k \quad \mathsf{N}$ *and any partition* $q^k\mathsf{N} = \bigcup_{i=1}^{v} A_i$*, there exists* $1 \leq i_0 \leq v$ *and* $y_1, \cdots, y_n \quad A_{i_0}$ *for which* $P(y_1, \cdots, y_n) = 0$.

Theorem 6.5.12 (cf. Theorem 3.3 of [BLM21]). *Suppose that* $p \quad \mathsf{N}$ *is a prime. If* P $\mathsf{Z}[x_1, \cdots, x_n]$ *is* p*-partition regular, then there exists a lower Rado functional* $(J_0, \cdots, J_\ell, d_1, \cdots, d_m)$ *for* P *such that setting* $d_0 = 0$*, the equation*

$$\sum_{i=0}^{m} p^{d_i} \sum_{\alpha \quad J_i} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(0, 0, \cdots, 0) w^{|\alpha|} = 0 \tag{6.76}$$

has an invertible solution in the ring Z_p *of* p*-adic integers.*

We provide a lemma on a condition in order for our polynomial to be partition regular.

Lemma 6.5.13. *Suppose that* $a, b, c \quad \mathsf{Z} \setminus \{0\}$ *and* $n \quad \mathsf{N}$ *are such that the equation*

$$ax + by = cwz^n \tag{6.77}$$

is partition regular. If p *is a prime for which* $v_p(\frac{a+b}{c}) \ / \ n\mathsf{N} \quad \{0\}$*, then equation* (6.77) *is* p*-partition regular.*

Proof. We will use induction on k to show that equation (6.77) is partition regular over $p^k\mathsf{N}$ for each $k \geq 0$. The base case of $k = 0$ holds by assumption, so let us proceed to the inductive step and show that the desired result holds for $k + 1$ if it holds for k. Let $m_1 = v_p(a + b)$, $m_2 = v_p(c)$, and $M = \max(m_1, m_2)$. Consider the partition

$$p^k\mathsf{N} = \bigcup_{j=1}^{p^{M+1}} C_j \text{ where } C_j = \{n \quad \mathsf{N} \mid n \equiv p^k j \pmod{p^{k+M+1}}\}. \tag{6.78}$$

Since equation (6.77) is partition regular over $p^k\mathsf{N}$, let $w, x, y, z \quad C_{j_0}$ satisfy equation (6.77). We see that

$$ax + by = cwz^n \rightarrow ax + by \equiv cwz^n \pmod{p^{k+M+1}} \tag{6.79}$$

$$= \quad aj_0 + bj_0 \equiv cj_0^{n+1}p^{nk} \equiv 0 \pmod{p^{M+1}}, \tag{6.80}$$

$$= \quad j_0(a + b - cj_0^n p^{nk}) \equiv 0 \pmod{p^{M+1}}. \tag{6.81}$$

Since $v_p(\frac{a+b}{c}) \ / \ n\mathsf{N} \quad \{0\}$, we see that $v_p(a+b-cj_0^n p^{nk}) = \min(v_p(a+b), v_p(cj_0^n p^{nk})) \leq M$, so we must have that $j_0 \equiv 0 \pmod{p}$. We now see that for any partition

$$p^{k+1}\mathsf{N} = \bigcup_{j=1}^{r} B_j, \tag{6.82}$$

we may let $B_{r+1} = p^k\mathbb{N} \setminus p^{k+1}\mathbb{N}$ and construct the partition

$$p^k\mathbb{N} = \bigcup_{\substack{1 \le j_1 \le r+1 \\ 1 \le j_2 \le p^{M+1}}} (B_{j_1} \cap C_{j_2}). \tag{6.83}$$

Since equation (6.77) is partition regular over $p^k\mathbb{N}$, let $w, x, y, z \in B_{J_1} \cap C_{J_2}$ satisfy equation (6.77). We have already shown that since $w, x, y, z \in C_{J_2}$, we have $w, x, y, z \in p^{k+1}\mathbb{N}$. Since $w, x, y, z \notin B_{r+1}$, we have shown that equation (6.77) is also partition regular over $p^{k+1}\mathbb{N}$ as desired. □

This leads us to the following useful criterion.

Theorem 6.5.14. *Suppose that $a, b, c \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$ are such that the equation*

$$ax + by = cwz^n \tag{6.84}$$

is partition regular. If p is a prime for which $v_p(\frac{a+b}{c}) \notin n\mathbb{N} \cup \{0\}$, then one of $\frac{a}{c}, \frac{b}{c}, \frac{a+b}{c}$ must be an nth power in $\mathbb{Q}_p$.

Proof. We will begin by determining all of the lower Rado functionals for $P(x_1, x_2, x_3, x_4) = ax_1 + bx_2 - cx_3x_4^n$. Since the system of equations

$$\begin{aligned} \phi(1,0,0,0) &= \alpha_1 = \alpha_2 = \phi(0,1,0,0) \\ \phi(0,1,0,0) &= \alpha_2 = \alpha_3 + n\alpha_4 = \phi(0,0,1,n) \end{aligned} \tag{6.85}$$

is partition regular, we see that

$$\Big\{\Big(\{(1,0,0,0),(0,1,0,0),(0,0,1,n)\}\Big), \Big(\{(1,0,0,0),(0,1,0,0)\},\{(0,0,1,n)\}\Big), \Big(\{(0,1,0,0),(0,0,1,n)\},\{(1,0,0,0)\}\Big), \Big(\{(1,0,0,0),(0,0,1,n)\},\{(0,1,0,0)\}\Big)\Big\},$$

is the collection of nontrivial[2] lower Rado functionals of order 0. We now proceed to determine all lower Rado functionals of order 1. Let $(J_0, \cdots, J_\ell, d_1)$ be a lower Rado functional of order 1. Since $d_1 > 0$ we may use Theorem 6.2.5 to see that neither of the equations

$$d_1 = \phi(1,0,0,0) - \phi(0,1,0,0) = \alpha_1 - \alpha_2, \text{ and} \tag{6.86}$$

$$d_1 = \phi(0,1,0,0) - \phi(1,0,0,0) = \alpha_2 - \alpha_1 \tag{6.87}$$

[2] A lower Rado functional or order 0 $(J_0, \cdots, J_\ell)$ is trivial if J_0 is a singleton, as such a functional will never yield an invertible solution to equation (6.76).

are partition regular. It follows that any lower Rado functional of order 1 must have $(0,0,1,n)$ J_0 J_1. We also note by Theorem 6.2.5 that the equations

$$d_1 = \phi(1,0,0,0) - \phi(0,0,1,n) = \alpha_1 - \alpha_3 - n\alpha_4, \tag{6.88}$$

$$d_1 = \phi(0,1,0,0) - \phi(0,0,1,n) = \alpha_2 - \alpha_3 - n\alpha_4, \tag{6.89}$$

$$d_1 = \phi(0,0,1,n) - \phi(1,0,0,0) = -\alpha_1 + \alpha_3 + n\alpha_4 \tag{6.90}$$

$$d_1 = \phi(0,0,1,n) - \phi(0,1,0,0) = -\alpha_2 + \alpha_3 + n\alpha_4, \tag{6.91}$$

are partition regular over $\mathbb{Z}$ if and only if $n|d_1$. This results in the following list of lower Rado functionals of order 1:

$$\Big\{\Big(\{(1,0,0,0)\},\{(0,0,1,n)\},\{(0,1,0,0)\},nd\Big),\Big(\{(0,1,0,0)\},\{(0,0,1,n)\},\{(1,0,0,0)\},nd\Big),$$
$$\Big(\{(0,0,1,n)\},\{(1,0,0,0)\},\{(0,1,0,0)\},nd\Big),\Big(\{(0,0,1,n)\},\{(0,1,0,0\},\{(1,0,0,0)\},nd\Big),$$
$$\Big(\{(1,0,0,0),(0,1,0,0)\},\{(0,0,1,n)\},nd\Big),\Big(\{(0,0,1,n)\},\{(1,0,0,0),(0,1,0,0)\},nd\Big)\Big\}.$$

Lastly, we recall that the only lower Rado functionals of order 2 are of the form $(J_0, J_1, J_2, d_1, d_2)$, but such a lower Rado functional cannot exist since equations (6.86) and (6.87) are not partition regular. We have now determined all of the lower Rado functionals for $P(x_1, x_2, x_3, x_4)$. By Lemma 6.5.13 we see that equation (6.84) is p-partition regular, so we may apply Theorem 6.5.12 to see that at least 1 of equations (6.92)-(6.101) has an invertible solution in

Z_p.

$$(a+b)w - cw^n = 0\Big(\text{from the lower Rado functional}\Big(\{(1,0,0,0),(0,1,0,0),(0,0,1,n)\}\Big)\Big), \tag{6.92}$$

$$(a+b)w - cp^{nd}w^n = 0\Big(\text{from the LRF}\Big(\{(1,0,0,0),(0,1,0,0)\},\{(0,0,1,n)\},nd\Big)\Big), \tag{6.93}$$

$$(a+b)p^{nd}w - cw^n = 0\Big(\text{from the LRF}\Big(\{(0,0,1,n)\},\{(1,0,0,0),(0,1,0,0)\},nd\Big)\Big), \tag{6.94}$$

$$aw - cw^n = 0\Big(\text{from the LRF}\Big(\{(1,0,0,0),(0,0,1,n)\},\{(0,1,0,0)\}\Big)\Big), \tag{6.95}$$

$$aw - cp^{nd}w = 0\Big(\text{from the LRF}\Big(\{(1,0,0,0)\},\{(0,0,1,n)\},\{(0,1,0,0)\},nd\Big)\Big), \tag{6.96}$$

$$ap^{nd}w - cw^n = 0\Big(\text{from the LRF}\Big(\{(0,0,1,n)\},\{(1,0,0,0)\},\{(0,1,0,0)\},nd\Big)\Big), \tag{6.97}$$

$$bw - cw^n = 0\Big(\text{from the LRF}\Big(\{(0,1,0,0),(0,0,1,n)\},\{(1,0,0,0)\}\Big)\Big), \tag{6.98}$$

$$bw - cp^{nd}w = 0\Big(\text{from the LRF}\Big(\{(0,1,0,0)\},\{(0,0,1,n)\},\{(1,0,0,0)\},nd\Big)\Big), \tag{6.99}$$

$$bp^{nd}w - cw^n = 0\Big(\text{from the LRF}\Big(\{(0,0,1,n)\},\{(0,1,0,0\},\{(1,0,0,0)\},nd\Big)\Big), \tag{6.100}$$

$$(a+b)w = 0\Big(\text{from the LRF}\Big(\{(1,0,0,0),(0,1,0,0)\},\{(0,0,1,n)\}\Big)\Big). \tag{6.101}$$

The desired result follows after noting that one of equations (6.92)-(6.101) has an invertible solution in Z_p if and only if one of $\frac{a}{c}, \frac{b}{c}, \frac{a+b}{c}$ is an nth power in Q_p. □

Before using Theorem 6.5.14, let us recall when a Z_2 is an nth power. If $a = 2^k m$ with m odd, it is a well-known consequence of Hensel's lemma that a is a 2^nth power in Z_2 if and only if $2^n \mid k$ and $m \equiv 1 \pmod{2^{n+2}}$.

Corollary 6.5.15. *The following equations are not partition regular as seen by an application of Theorem 6.5.14 with $p = 2$ for items (i)-(iii), $p = 3$ for item (iv), and $p = 5$ for item (v).*

(i) $3x + 13y = wz^8$.

Observe that 16 is an 8th power modulo p for every prime p, so this equation is not susceptible to Theorem 6.3.8.

(ii) $16x + 16y = wz^8$.

Observe that 16 *is an* 8*th power modulo* p *for every prime* p*, so this equation is not susceptible to Theorem 6.3.8.*

(iii) $3 \cdot 5 \cdot 2^2 x + 2 \cdot 5 \cdot 3^2 y = wz^2$.

Observe that at least one of $3 \cdot 5 \cdot 2^2, 2 \cdot 5 \cdot 3^2$*, or* $3 \cdot 5 \cdot 2^2 + 2 \cdot 5 \cdot 3^2 = 2 \cdot 3 \cdot 5^2$ *is a square modulo* p *for every prime* p*, so this equation is not susceptible to Theorem 6.3.8.*

(iv) $3^4 x + 3^6 y = wz^{12}$.

Observe that -3 *is a square modulo* p *if* $p \equiv 1 \pmod 3$ *and* 3 *is a cube modulo* p *if* $p \equiv 2 \pmod 3$*, so either* 3^4 *or* 3^6 *is a* 12*th power modulo any prime* p *(cf. Item (ii)(a) or Remark 6.4.3), so this equation is not susceptible to Theorem 6.3.8.*

(v) $(3^2 \cdot 4 \cdot 5)^2 x + (3 \cdot 4^2 \cdot 5)^2 y = wz^4$

Observe that $\alpha = (3^2 \cdot 4 \cdot 5)^2, \beta = (3 \cdot 4^2 \cdot 5)^2$*, and* $\gamma = \alpha + \beta = (3 \cdot 4 \cdot 5^2)^2$ *are all squares but are not fourth powers. Since one of* $3^2 \cdot 4 \cdot 5, 3 \cdot 4^2 \cdot 5$*, and* $3 \cdot 4 \cdot 5^2$ *will be a perfect square modulo any prime* p*, we see that one of* α^2, β^2*, or* γ^2 *will be a perfect fourth power modulo any prime* p*, so this equation is not susceptible to Theorem 6.3.8.*

6.6 On the Partition Regularity of $ax + by = cwz^n$ over Integral Domains

The purpose of this section is to try and generalize as much of Theorem 6.1.1 as we can to the more general setting of integral domains instead of just $\mathbb{Z}$ or $\mathbb{N}$.

Theorem 6.6.1 (cf. Theorem 6.9.18)**.** *Let* R *be an integral domain. There exists an ultrafilter* $p \in \beta R \setminus \{0\}$ *with the following properties.*

(i) For any $A \in p$ *and* $\ell \in \mathbb{N}$*, there exists* $b, g \in A$ *with* $\{bg^j\}_{j=0}^{\ell} \subseteq A$*.*

(ii) For any $A \in p$*,* $\ell \in \mathbb{N}$*, and* $h, s \in R \setminus \{0\}$*, there exists* $a, d \in R$ *for which* $\{hd, ha, ha + sd\} \subseteq A$*.*

(iii) For every $\alpha \in R \setminus \{0\}$*, we have* $\alpha R \in p$*.*

Lemma 6.6.2. *Let* R *be an integral domain and let* $p \in \beta R \setminus \{0\}$ *be an ultrafilter satisfying the conditions of Theorem 6.6.1. For any* $A \in p$*,* $a, b \in R \setminus \{0\}$ *and* $n \in \mathbb{N}$*, the equation*

$$ax + by = cwz^n \tag{6.102}$$

has a solution in A *if* $c \in \{a, b, a + b\}$*.*

Proof. Let

$$A \quad = \{v \quad A \mid v = wz^n \text{ for some } z, w \quad A\}. \tag{6.103}$$

Since $A \quad p$, to see that $A \quad = A \setminus (A \setminus A \ ) \quad p$ it suffices to observe that $A \setminus A \ \ / \ p$ because $A \setminus A$ does not satisfy condition (i) of Theorem 6.6.1. Our first case is when $c = a + b$, and in this case we let $x \quad A$ be arbitrary and let $w, z \quad A$ be such that $x = wz^n$. Since

$$ax + bx = cx = cwz^n, \tag{6.104}$$

we see that x, x, w, z is a solution to equation (6.102) coming from A. For our second case it suffices to consider $c = a$ since the case of $c = b$ is handled similarly. Observe that $A_a := A \ \cap aR \quad p$ since $A \ , aR \quad p$ and consider

$$A \quad = \{x_1 \quad A_a \mid \text{there exists } x_2 \quad A_a \text{ satisfying } x_1 + b\frac{x_2}{a} \quad A_a\}. \tag{6.105}$$

Since $A_a \quad p$, to see that $A \quad = A_a \setminus (A_a \setminus A \ ) \quad p$ it suffices to observe that $A_a \setminus A \ \ / \ p$ because $A_a \setminus A$ does not satisfy condition (ii) of Theorem 6.6.1 with $(h, s) = (a, b)$. Now let $x_1 \quad A$ be arbitrary, let $x_2 \quad A_a$ be as in equation (6.105), and observe that

$$ax_1 + bx_2 = a(x_1 + b\frac{x_2}{a}). \tag{6.106}$$

Since $x_1 + b\frac{x_2}{a} \quad A$, we may pick $w, z \quad A$ for which $x_1 + b\frac{x_2}{a} = wz^n$. In this case we observe that

$$ax_1 + bx_2 = c(x_1 + b\frac{x_2}{a}) = cwz^n, \tag{6.107}$$

so x_1, x_2, w, z is a solution to equation (6.102) coming from A. □

Before proceeding further let us recall some notation. If R is an integral domain, then for $u, v \quad R \setminus \{0\}$ and $A \quad R$ we have

$$\frac{v}{u}A = \{r \quad R \mid \frac{u}{v}r \quad A\} = \{\frac{v}{u}a \mid a \quad A \cap uR\}. \tag{6.108}$$

Similarly, if $p \quad \beta R$ is an ultrafilter, then we have

$$\frac{u}{v} \cdot p = \{A \quad R \mid \frac{v}{u}A \quad p\} = \{\frac{u}{v}A \mid A \quad p\}. \tag{6.109}$$

Theorem 6.6.3. *Let R be an integral domain with field of fractions K and let $p \in \beta R \setminus \{0\}$ be an ultrafilter satisfying the conditions of Theorem 6.6.1. If $a, b, c \in R \setminus \{0\}$ and $n \in \mathbb{N}$ are such that one of $\frac{a}{c}$, $\frac{b}{c}$, or $\frac{a+b}{c}$ is of the form $(\frac{u}{v})^n$ for some $u, v \in R$, then the equation*

$$ax + by = cwz^n \tag{6.110}$$

contains a solution for any $A \in q$ where

$$q = \begin{cases} p & \text{if } u = 0 \\ \frac{u}{v} \cdot p & \text{else} \end{cases}. \tag{6.111}$$

In particular, equation (6.110) is partition regular over $R \setminus \{0\}$.

Proof. We see that if $u = 0$ then $a + b = 0$, so the desired result in this case follows from Lemma 6.6.2. Now let us assume that $u = 0$ and let $d \in \{a, b, a+b\}$ be such that $\frac{d}{c} = (\frac{u}{v})^n$. Let $A \in \frac{u}{v} \cdot p$ be arbitrary and note that $\frac{v}{u}A \in p$. By Lemma 6.6.2 there exists $w, x, y, z \in A$ for which $\frac{v}{u}w, \frac{v}{u}x, \frac{v}{u}y, \frac{v}{u}z \in \frac{v}{u}A$ and

$$a(\frac{v}{u}x) + b(\frac{v}{u}y) = d(\frac{v}{u}w)(\frac{v}{u}z)^n \quad \Rightarrow \quad ax + by = d(\frac{v}{u})^n wz^n = cwz^n. \tag{6.112}$$

For the latter half of the Theorem, it suffices to note that if $R \setminus \{0\} = \bigcup_{i=1}^r C_i$ is a partition, then there exists $1 \leq i_0 \leq r$ for which $C_{i_0} \in q$, hence C_{i_0} contains a solution to equation (6.110). □

We recall that if R is an Dedekind domain and $\mathfrak{p} \subseteq R$ is a prime (hence maximal) ideal, then for any $u \in R$ and $v \in R \setminus \{0\}$ we have $\frac{u}{v} \equiv uv^{-1} \pmod{\mathfrak{p}}$.

Theorem 6.6.4. *Let R be a Dedekind domain with field of fractions K. Let $a, b, c \in R \setminus \{0\}$ and $n \in \mathbb{N}$ be such that none of $\frac{a}{c}$, $\frac{b}{c}$, or $\frac{a+b}{c}$ are nth powers in $R/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq R$ satisfying $a, b, a+b, c \notin \mathfrak{p}$ and $[R : \mathfrak{p}] < \infty$. Let $K_\mathfrak{p}$ denote the completion of K at $\mathfrak{p}$. The equation*

$$ax + by = cwz^n \tag{6.113}$$

is not partition regular over $K_\mathfrak{p} \setminus \{0\}$.

Proof. Since R is a Dedekind domain we see that $R_\mathfrak{p}$ is a discrete valuation ring under the valuation $v_\mathfrak{p}$, so let π be a generator of the maximal ideal of $R_\mathfrak{p}$. Let $F \subseteq R$ be a set of coset representatives of (π) such that $\bigcup_{f \in F}(f + (\pi)) = R \setminus \mathfrak{p}$ and $(f_1 + (\pi)) \cap (f_2 + (\pi)) = \emptyset$ whenever $f_1 \neq f_2$. We note that $|F| = [R : \mathfrak{p}] - 1 < \infty$. Let $\chi : K_\mathfrak{p} \setminus \{0\} \to F$ be given by

$$\frac{x}{\pi^{v_{\mathfrak{p}}(x)}} \equiv \chi(x) \pmod{\mathfrak{p}}. \tag{6.114}$$

Note that $\chi(rs) = \chi(r)\chi(s) \pmod{\pi}$ for all $r, s \quad R \setminus \{0\}$. We also see that

$$\chi(r+s) \equiv \begin{cases} \chi(r) + \chi(s) \pmod{\pi} & \text{if } v_{\mathfrak{p}}(r) = v_{\mathfrak{p}}(s) \text{ and } r + s \equiv 0 \pmod{\pi} \\ \chi(s) \pmod{\pi} & \text{if } v_{\mathfrak{p}}(r) > v_{\mathfrak{p}}(s) \\ \chi(r) \pmod{\pi} & \text{if } v_{\mathfrak{p}}(s) > v_{\mathfrak{p}}(r) \end{cases}. \tag{6.115}$$

Let $K_{\mathfrak{p}} \setminus \{0\} = \bigcup_{f \quad F} C_f$ be the partition given by $C_f = \chi^{-1}(\{f\})$. Let us assume for the sake of contradiction that there exists $d \quad F$ for which $w, x, y, z \quad C_d$ and equation (6.113) is satisfied. We now have 3 cases to consider. If $v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}(y)$, then we see that

$$0 \equiv (a+b)d \equiv \chi(a)\chi(x) + \chi(b)\chi(y) \equiv \chi(ax+by) \equiv \chi(cwz^n) \equiv cd^{n+1} \pmod{\mathfrak{p}} \tag{6.116}$$

$$= \quad (a+b)c^{-1} \equiv d^n \pmod{\mathfrak{p}}, \tag{6.117}$$

which yields the desired contradiction. For our next case we assume that $v_{\mathfrak{p}}(x) < v_{\mathfrak{p}}(y)$ and note that

$$0 \equiv ad \equiv \chi(a)\chi(x) \equiv \chi(ax+by) \equiv \chi(cwz^n) \equiv cd^{n+1} \pmod{\mathfrak{p}} \tag{6.118}$$

$$= \quad ac^{-1} \equiv d^n \pmod{\mathfrak{p}}, \tag{6.119}$$

which again yields the desired contradiction. Similarly, in our final case when $v_{\mathfrak{p}}(x) > v_{\mathfrak{p}}(y)$ we have

$$0 \equiv bd \equiv \chi(b)\chi(y) \equiv \chi(ax+by) \equiv \chi(cwz^n) \equiv cd^{n+1} \pmod{\mathfrak{p}} \tag{6.120}$$

$$= \quad bc^{-1} \equiv d^n \pmod{\mathfrak{p}}, \tag{6.121}$$

which once more yields the desired contradiction. □

Corollary 6.6.5. *Let K be a number field and let ω_m be the number of mth roots of unity in K. Let $a, b, c \in O_K$ and let $n \in \mathbb{N}$. Let d_a be the largest integer for which $a^{\frac{1}{d_a}} \in O_K$, and define d_b and d_c similarly. Let $m_a = \frac{n}{d_a}, m_b = \frac{n}{d_b}$, and $m_c = \frac{n}{d_c}$.*

(i) Suppose n is odd, and none of $a^{\omega_{m_a}}, b^{\omega_{m_b}}$, and $c^{\omega_{m_c}}$ are an nth power in O_K; or

(ii) Suppose n is even, and a, b, c satisfy the same conditions as in (i), but $a^{2\omega_{m_a}}$ is also not an nth power.

Then the equation

$$ax + by = cwz^n \tag{6.122}$$

is not partition regular over $K \setminus \{0\}$.

Proof. The given assumptions are precisely what we need to use Lemma 6.4.7 and obtain a prime ideal $\mathfrak{p} \in O_K$ for which none of a, b, and c are nth powers modulo $\mathfrak{p}$. After noting that K embeds in $K_{\mathfrak{p}}$, we see that the desired result follows from Theorem 6.6.4. □

Remark 6.6.6. Consider the equation

$$2x + 3y = wz^2. \tag{6.123}$$

Since $2, 3$, and 5 are not squares modulo 43, Theorem 6.6.5 tells us that equation (6.123) is not partition regular over $\mathbb{Q}_{43} \setminus \{0\}$. Since equation (6.123) is partition regular over the countable set $\mathbb{Z}[\sqrt{2}]$ as a consequence of Theorem 6.6.3 but not over the uncountable set $\mathbb{Q}_{43}$, we see that the algebraic properties of the underlying set S have a stronger influence on the partition regularity of equations of the form $ax + by = cwz^n$ than the cardinality of S.

6.7 Systems of Equations

Theorem 6.7.1. *Let R be an integral domain with field of fractions K and let $p \in \beta R \setminus \{0\}$ be an ultrafilter satisfying the conditions of Theorem 6.6.1. If $a_1, \cdots, a_k, b_1, \cdots, b_k, c_1, \cdots, c_k \in R \setminus \{0\}$ and $n_1, \cdots, n_k \in \mathbb{N}$ are such that*

$$I := \bigcap_{i=1}^{k} \left\{ \sqrt[n_i]{\frac{a_i}{c_i}}, \sqrt[n_i]{\frac{b_i}{c_i}}, \sqrt[n_i]{\frac{a_i + b_i}{c_i}} \right\} = \quad , \tag{6.124}$$

then the system of equations

$$\begin{aligned} a_1x_1 &+ b_1y_1 &= c_1w_1z_1^n \\ a_2x_2 &+ b_2y_2 &= c_2w_2z_2^n \\ &\vdots \\ a_kx_k &+ b_ky_k &= c_kw_kz_k^n \end{aligned} \tag{6.125}$$

contains a solution in every $A \quad q$, *where we may take*

$$q = \begin{cases} p & \text{if } 0 \quad I \\ i \cdot p & \text{if there exists } i \quad I \setminus \{0\} \end{cases}. \tag{6.126}$$

In particular, the system of equations in (6.125) *is partition regular over* $R \setminus \{0\}$.

Proof. Since none of the equations in the system of equations in (6.125) share any variables, the desired result follows from Theorem 6.6.3 □

Theorem 6.7.2. *Let* R *be a Dedekind domain with field of fractions* K. *Let* $a_1, \cdots, a_k, b_1, \cdots, b_k, c_1, \cdots, c_k$ $R \setminus \{0\}$, *let* $n \quad \mathbb{N}$, *and let*

$$I := \bigcap_{i=1}^{k} \left\{ \frac{a_i}{c_i}, \frac{b_i}{c_i}, \frac{a_i + b_i}{c_i} \right\} \tag{6.127}$$

Suppose that there exists a prime ideal $\mathfrak{p} \quad R$ *satisfying:*

(i) $a_1, \cdots, a_i, b_1, \cdots, b_i, c_1, \cdots, c_i, a_1 + b_1, \cdots, a_i + b_i \ / \ \mathfrak{p}$.

(ii) If $v_1, v_2 \quad \bigcup_{i=1}^{k} \{\frac{a_i}{c_i}, \frac{b_i}{c_i}, \frac{a_i+b_i}{c_i}\}$ *are distinct, then* $v_1 \equiv v_2 \pmod{\mathfrak{p}}$.

(iii) No element of I *is an* n*th power modulo* $\mathfrak{p}$.

(iv) $[R : \mathfrak{p}] < \infty$.

The system of equations

$$\begin{aligned} a_1x_1 &+ b_1y_1 &= c_1w_1z_1^n \\ a_2x_2 &+ b_2y_2 &= c_2w_2z_2^n \\ &\vdots \\ a_kx_k &+ b_ky_k &= c_kw_kz_k^n \end{aligned} \tag{6.128}$$

is not partition regular over $K_{\mathfrak{p}} \setminus \{0\}$, *where* $K_{\mathfrak{p}}$ *is the localization of* K *at* $\mathfrak{p}$.

Proof. We begin the proof similarly to that of Theorem 6.6.4. Since R is a Dedekind domain we see that $R_{\mathfrak{p}}$ is a discrete valuation ring under the valuation $v_{\mathfrak{p}}$, so let π be a generator of the maximal ideal of $R_{\mathfrak{p}}$. Let $F \quad R$ be a set of coset representatives of (π) such that $\bigcup_{f \quad F}(f+(\pi)) = R \setminus \mathfrak{p}$ and $(f_1+(\pi)) \cap (f_2+(\pi)) = \quad$ whenever $f_1 = f_2$. We note that $|F| = [R:\mathfrak{p}] - 1 < \infty$. Let $\chi : K_{\mathfrak{p}} \setminus \{0\} \to F$ be given by

$$\frac{x}{\pi^{v_{\mathfrak{p}}(x)}} \equiv \chi(x) \pmod{\mathfrak{p}}. \tag{6.129}$$

Observe that $\chi(r) \equiv r \pmod{\mathfrak{p}}$ for all $r \,/\, \mathfrak{p}$. Let $K_{\mathfrak{p}} \setminus \{0\} = \bigcup_{f \quad F} C_f$ be the partition given by $C_f = \chi^{-1}(\{f\})$. Let us assume for the sake of contradiction that there exists $d \quad F$ and $x_1, x_2, y_1, y_2, w_1, w_2, z_1, z_2 \quad C_d$ for which the system of equations in (6.128) is satisfied. We see that for $1 \le i \le k$ we have

$$0 \equiv \chi(c_i w_i z_i^n) \equiv \chi(a_i x_i + b_i y_i) \equiv \begin{cases} \chi(a_i)d + \chi(b_i)d & \pmod{\pi} & \text{if } v_{\mathfrak{p}}(x_i) = v_{\mathfrak{p}}(y_i) \\ \chi(a_i)d & \pmod{\pi} & \text{if } v_{\mathfrak{p}}(y_i) > v_{\mathfrak{p}}(x_i) \\ \chi(b_i)d & \pmod{\pi} & \text{if } v_{\mathfrak{p}}(x_i) > v_{\mathfrak{p}}(x_i) \end{cases} \tag{6.130}$$

$$d^n \equiv d^{-1}\chi(w_i z_i^n) = \begin{cases} (a_i+b_i)c^{-1} & \pmod{\pi} & \text{if } v_{\mathfrak{p}}(x_i) = v_{\mathfrak{p}}(y_i) \\ a_i c^{-1} & \pmod{\pi} & \text{if } v_{\mathfrak{p}}(y_i) > v_{\mathfrak{p}}(x_i) \\ b_i c^{-1} & \pmod{\pi} & \text{if } v_{\mathfrak{p}}(x_i) > v_{\mathfrak{p}}(x_i) \end{cases}. \tag{6.131}$$

For $1 \le i \le k$ let $v_i \quad \{a_i c_i^{-1}, b_i c_i^{-1}, (a_i+b_i)c_i^{-1}\}$ be such that $d^n \equiv v_i \pmod{\mathfrak{p}}$. Since we must have that $v_i \equiv d^n \equiv v_j \pmod{\mathfrak{p}}$ for all $1 \le i < j \le k$, we see that there is some $v \quad K$ for which $v = v_i$ for all $1 \le i \le k$, hence $v \quad I$. The desired contradictions follows after recalling that no element of I is an nth power modulo $\mathfrak{p}$. □

We observe that the conditions of Theorem 6.7.2 are vacuously fulfilled if $I = \quad$.

Corollary 6.7.3. *Let* $a_1, \cdots, a_k, b_1, \cdots, b_k, c_1, \cdots, c_k \quad \mathbb{Z} \setminus \{0\}$ *and* $n \quad \mathbb{N}$ *be such that the system of equations*

$$\begin{array}{ccccc} a_1x_1 & + & b_1y_1 & = & c_1w_1z_1^n \\ a_2x_2 & + & b_2y_2 & = & c_2w_2z_2^n \\ & & \vdots & & \\ a_kx_k & + & b_ky_k & = & c_kw_kz_k^n \end{array} \tag{6.132}$$

is partition regular over $\mathbb{Z} \setminus \{0\}$*. Let*

$$I := \bigcap_{i=1}^{k} \left\{ \frac{a_i}{c_i}, \frac{b_i}{c_i}, \frac{a_i + b_i}{c_i} \right\}. \tag{6.133}$$

(i) If $4 \nmid n$ then I contains an nth power.

(ii) If $4|n$ then I contains an $\frac{n}{2}$th power.

Proof. Let us assume for the sake of contradiction that one of items (i) and (ii) were false. Since $|I| \leq 2$, we may invoke Lemma 6.4.10 to find a prime p satisfying the conditions of Theorem 6.7.2 to attain the desired contradiction. □

Remark 6.7.4. The following systems of equations are not partition regular as seen by an application of Corollary 6.7.3.

(i)

$$\begin{aligned} 3^4 \cdot 4^2 \cdot 5^2 x_1 &+ 3^2 \cdot 4^4 \cdot 5^2 y_1 &= w_1 z_1^4 \\ 5^4 \cdot 12^2 \cdot 13^2 x_2 &+ 5^2 \cdot 12^4 \cdot 13^2 y_2 &= w_2 z_2^4 \end{aligned}, n \in \mathbb{N}$$

.

(ii)

$$\begin{aligned} 16x_1 &+ 17y_1 &= w_1 z_1^8 \\ 33x_2 &+ (2^{12} - 33)y_2 &= w_2 z_2^8 \end{aligned}$$

.

(iii)

$$\begin{aligned} 2^n x_1 &+ 3^n y_1 &= w_1 z_1^n \\ 3^n x_2 &+ 7^n y_2 &= w_2 z_2^n \\ 7^n x_3 &+ 2^n y_3 &= w_3 z_3^n \end{aligned}, n \in \mathbb{N}$$

.

(iv)

$$\begin{aligned} 9x_1 &+ 16y_1 &= w_1 z_1^2 \\ 25x_2 &- 9y_2 &= w_2 z_2^2 \\ 25x_3 &- 16y_3 &= w_3 z_3^2 \\ 9x_4 &+ 7y_4 &= w_4 z_4^2 \end{aligned}$$

.

A few remarks are in order regarding these examples. In example (i) neither of the constituent equations of the system are individually partition regular. This fact can be determined through the use of Lemma 6.5.14, but not through the use of Theorem 6.3.8 alone, despite the similarity of the proofs of Theorem 6.3.8 and 6.7.2. In example (ii) we do not currently know whether either of the constituent equations of the system are individually partition regular (cf. Section 6.8). In example (iii) any proper subsystem of equations is partition regular as a consequence of Theorem 6.7.1. Theorem 6.7.1 also shows us that in example (iv) the system of equations becomes partition regular if any equation other than the first equation is removed from the system.

6.8 Conjectures and Concluding Remarks

Theorem 6.1.1 and Corollary 6.5.15 naturally lead us to the following conjecture:

Conjecture 6.8.1. *Given* $a, b, c \in \mathbb{Z} \setminus \{0\}$ *and* $n \in \mathbb{N}$*, the equation*

$$ax + by = cwz^n \tag{6.134}$$

is partition regular over $\mathbb{Z} \setminus \{0\}$ *if and only if one of* $\frac{a}{c}, \frac{b}{c}, \frac{a+b}{c}$ *is an* n*th power in* $\mathbb{Q}$*.*

We see that the situation in which we have yet to resolve Conjecture 6.8.1 fully is when n is even and one of $\frac{a}{c}, \frac{b}{c}$, or $\frac{a+b}{c}$ is an $\frac{n}{2}$th power in $\mathbb{Q}$. Since some special instances of this situation have been resolved in Corollary 6.5.15, we list here some equations whose partition regularity remains unknown. Firstly, the equation

$$33x + (2^{12} - 33)y = wz^8 \tag{6.135}$$

is not expected to be partition regular since $33, 2^{12} - 33$, and 2^{12} are not 8th powers, but 2^{12} is an 8th power modulo p for every prime p, and 33 is an 8th power in $\mathbb{Z}_2$, so we are unable to apply Theorem 6.3.8 or Theorem 6.5.14. Similarly, the equation

$$16x + 17y = wz^8 \tag{6.136}$$

is not expected to be partition regular since $16, 17$, and 33 are not 8th powers, but 16 is an 8th power modulo p for every prime p, and 16 is also an 8th power in $\mathbb{Z}_3$ and $\mathbb{Z}_{11}$, so we are once again unable to apply Theorem 6.3.8 or Theorem 6.5.14. Next, we see that for any coprime $a, b \in \mathbb{N}$ for which a, b, and $a + b$ are not squares, the equation

$$a^2b(a + b)x + ab^2(a + b)y = wz^2 \tag{6.137}$$

is not expected to be partition regular since none of $a^2b(a + b), ab^2(a + b)$, and $a^2b(a + b) + ab^2(a + b) = ab(a + b)^2$ are squares. However, at least one of $a^2b(a + b), ab^2(a + b)$, and

$ab(a+b)^2$ is a square modulo p for any prime p, so we cannot make use of Theorem 6.3.8. We have seen in item (iii) of Corollary 6.5.15 that Theorem 6.5.14 can be used in some cases, but it is unclear to the authors as to whether or not Lemma 6.5.14 can be used in all cases. A similar difficulty arises in the case of $n = 4$. Recalling that for any $m, n, k \in \mathbb{N}$ we have $(2mnk)^2 + (km^2 - kn^2)^2 = (km^2 + kn^2)^2$, we take $k = 2mn(m^2 - n^2)(m^2 + n^2)$ and consider the equation

$$\Big((2mn)^2(m^2 - n^2)(m^2 + n^2)\Big)^2 x + \Big(2mn(m^2 - n^2)^2(m^2 + n^2)\Big)^2 y = wz^4. \tag{6.138}$$

We see that for any prime p at least one of $\alpha := (2mn)^2(m^2 - n^2)(m^2 + n^2), \beta := 2mn(m^2 - n^2)^2(m^2 + n^2)$, or $\gamma := 2mn(m^2 - n^2)(m^2 + n^2)^2$ will be a perfect square modulo p, hence one of α^2, β^2, or γ^2 will be a fourth power modulo p, so equation (6.138) is not susceptible to Theorem 6.3.8. We saw in item (v) of Corollary 6.5.15 that Lemma 6.5.14 can occasionally be of use in this situation, but it is once again unclear to the authors whether or not Lemma 6.5.14 can always be used in this situation.

While the methods of this paper are not strong enough to fully resolve Conjecture 6.8.1, they are strong enough to prove the following Theorem:

Theorem 6.8.2. *Fix* $a_1, \cdots, a_r, b_1, \cdots, b_s, c \in \mathbb{Z} \setminus \{0\}$.

(i) If $\min(b_1, \cdots, b_s) \geq 2$ *then the equation*

$$\sum_{i=1}^{r} a_i x_i = c \prod_{j=1}^{s} y_j^{b_j} \tag{6.139}$$

is partition regular over $\mathbb{Z} \setminus \{0\}$ *if and only if there exists* $F \subseteq [1, r]$ *for which* $\sum_{i \in F} a_i = 0$.

(ii) If $s \geq 2$, *then the equation*

$$\sum_{i=1}^{r} a_i x_i = c y_1 \prod_{j=2}^{s} y_j^{b_j} \tag{6.140}$$

is partition regular over $\mathbb{Z} \setminus \{0\}$ *if for some* $\emptyset \neq F \subseteq [1, r], s_F := \sum_{i \in F} \frac{a_i}{c}$ *is an* n*th power in* $\mathbb{Q}$, *where* $n = \sum_{j=2}^{s} b_j$. *Furthermore, if there exists* $F \subseteq [1, r]$ *for which* s_F *is a* n*th power in* $\mathbb{Q}_{\geq 0}$, *then equation* (6.140) *is partition regular over* $\mathbb{N}$.

(iii) The equation

$$\sum_{i=1}^{r} a_i x_i = c y_1 \prod_{j=2}^{s} y_j^{b_j} \tag{6.141}$$

is not partition regular over $\mathbb{Q} \setminus \{0\}$ *if there exists a prime* p *such that for any* $\emptyset \neq F \subseteq [1, r], \sum_{i \in F} \frac{a_i}{c}$ *is not an* n*th power modulo* p, *where* $n = \sum_{j=2}^{s} b_j$.

This naturally leads to the following generalization of Conjecture 6.8.1.

Conjecture 6.8.3. *Given* $a_1, \cdots, a_r, c \quad \mathbb{Z} \setminus \{0\}$ *and* $b_2, \cdots, b_s \quad \mathbb{N}$*, the equation*

$$\sum_{i=1}^{r} a_i x_i = c y_1 \prod_{j=2}^{s} y_j^{b_j} \tag{6.142}$$

is partition regular over $\mathbb{Z} \setminus \{0\}$ *if and only if there is some* $\quad = F \quad [1, r]$ *for which* $\sum_{i \; F} \frac{a_i}{c}$ *is an* n*th power in* $\mathbb{Q}$*, where* $n = \sum_{j=2}^{s} b_j$*.*

We have already seen that the methods of this paper cannot be extended to prove Conjecture 6.8.3 even when $r = 2$. When $r > 2$, there are even more problematic cases to consider. For example, if $r = 4$ then at least one of $2, 5, 10$, or 20 is a perfect cube modulo p for any prime p, so we are unable to use item (iii) of Theorem 6.8.2 to show that the equation

$$2x_1 + 5x_2 + 10x_3 + 20x_4 = y_1 y_2^3 \tag{6.143}$$

is not partition regular over $\mathbb{Z}$. Furthermore, since $7^3 \equiv 10 \pmod{37}$, we may use Hensel's lemma to see that 10 is a perfect cube in $\mathbb{Z}_{37}$, so analogues of Lemma 6.5.13 will be of no use here.

In light of Theorem 6.6.3 and Corollary 6.6.5 it is natural to pose the following question.

Question 6.8.4. *Given an integral domain* R*,* $r_1, \cdots, r_k, c \quad R \setminus \{0\}$*, and* $n_1, \cdots, n_s \quad \mathbb{N}$*, when is*

$$\sum_{i=1}^{k} r_i x_i = c \prod_{j=1}^{s} y_j^{b_j} \tag{6.144}$$

partition regular over $R \setminus \{0\}$*?*

An analog of Theorem 6.5.8 would have to be proven for polynomial equations over R instead of $\mathbb{N}$ in order to prove an analog of item (i) of Theorem 6.8.2, which would help partially answer Question 6.8.4.

Remark 6.8.5. In light of Remark 6.3.7 we are led to ask about sign obstructions to partition regularity of polynomial equations in rings of integers of totally real number fields. Let us consider for example the number field $K = \mathbb{Q}[\sqrt{2}]$, which is totally real since all of its embeddings into $\mathbb{C}$ turn out to be embeddings into $\mathbb{R}$. We recall that $O_K = \mathbb{Z}[\sqrt{2}]$ is the ring of integers of K. It is clear that the equation

$$x + y = -wz \tag{6.145}$$

is not partition regular regular over $\mathbb{Z}[\sqrt{2}]_{>0}$ due to sign obstructions despite being partition regular over $\mathbb{Z}[\sqrt{2}] \setminus \{0\}$ as a consequence of Theorem 6.6.3. We are unable to determine whether or not the equation

$$2x - 2\sqrt{2}y = wz^3 \tag{6.146}$$

is partition regular over $\mathbb{Z}[\sqrt{2}]_{>0}$ since there are no sign obstructions but we are unable to apply Theorem 6.6.3 and we are unable to apply the methods of Theorem 6.3.5 since $-2\sqrt{2} = (-\sqrt{2})^3$ and $-\sqrt{2} \notin \mathbb{Z}[\sqrt{2}]_{>0}$. Unlike Remark 6.3.7, we may take this line of inquiry a step further by examining the semiring P of totally positive elements of $\mathbb{Z}[\sqrt{2}]$, which are those elements that remain positive under every embedding of $\mathbb{Q}[\sqrt{2}]$ into $\mathbb{R}$. In this case we can directly determine P to be given by $P = \{a + b\sqrt{2} \mid a > |b|\sqrt{2}\}$. It is clear that equation (6.145) is not partition regular over P since $P \subseteq \mathbb{Z}[\sqrt{2}]_{>0}$, but can we determine whether or not equation (6.146) is partition regular over P? Furthermore, it can be shown using the techniques of this paper that the equation

$$\sqrt{2}x + 2\sqrt{2}y = wz^3 \tag{6.147}$$

is partition regular over $\mathbb{Z}[\sqrt{2}]_{>0}$ since $2\sqrt{2} = (\sqrt{2})^3$ and $\sqrt{2} \in \mathbb{Z}[\sqrt{2}]_{>0}$. However, $\sqrt{2}$ is a positive element of $\mathbb{Z}[\sqrt{2}]$ that is not totally positive, so equation (6.147) is not partition regular over P since $w, x, y, z \in P$ woud result in the left hand side of the equation being positive but not totally positive, while the right hand side of the equation would be totally positive.

Now let us consider the equation

$$2x + 2\sqrt{2}y = wz^3. \tag{6.148}$$

We can show that equation (6.148) is partition regular over $\mathbb{Z}[\sqrt{2}]_{>0}$ using the considerations from above, but now that there are no "generalized sign obstructions" can we also determine whether or not equation (6.148) is partition regular over P?

In light of Theorem 6.7.1 and Corollary 6.7.3 we are led to the Conjecture 6.8.6.

Conjecture 6.8.6 (cf. Conjecture 6.1.3). *Let* $a_1, \cdots, a_k, b_1, \cdots, b_k, c_1, \cdots, c_k \in \mathbb{Z} \setminus \{0\}$ *and* $n \in \mathbb{N}$. *The system of equations*

$$\begin{array}{ccccc} a_1x_1 & + & b_1y_1 & = & c_1w_1z_1^n \\ a_2x_2 & + & b_2y_2 & = & c_2w_2z_2^n \\ & & \vdots & & \\ a_kx_k & + & b_ky_k & = & c_kw_kz_k^n \end{array} \tag{6.149}$$

is partition regular over Z \ {0} *if and only if*

$$I := \bigcap_{i=1}^{k} \left\{ \frac{a_i}{c_i}, \frac{b_i}{c_i}, \frac{a_i + b_i}{c_i} \right\} \tag{6.150}$$

contains an n*th power in* Q*.*

We conclude with some examples of systems of equations whose partition regularity remains unknown. We are unable to apply Theorem 6.7.2 to the system of equations

$$\begin{array}{rcrcl} 16x_1 & + & 17y_1 & = & w_1 z_1^8 \\ 33x_2 & - & 17y_2 & = & w_2 z_2^8 \end{array} \tag{6.151}$$

since $I = \{16, 33\}$ and 16 is an 8th power modulo every prime p. Since 33 is an 8th power in Z_2 we also cannot expect a generalization of Lemma 6.5.14 to systems of equations to help determine the partition regularity of the system in (6.151). We are also unable to apply Theorem 6.7.2 to the system of equations

$$\begin{array}{rcrcl} 5^4 x_1 & + & 3^6 y_1 & = & w_1 z_1^{12} \\ (5^4 - 3^6)x_2 & + & 3^6 y_2 & = & w_2 z_2^{12} \end{array}, \tag{6.152}$$

since $I = \{5^4, 3^6\}$, 5 is a cube modulo p when $p \equiv 2 \pmod 3$, and -3 is a square modulo p when $p \equiv 1 \pmod 3$, so one of 5^4 and 3^6 will be a 12th power modulo any prime p (cf. Item (ii)(a) of Remark 6.4.3).

6.9 The Existence of Special Ultrafilters

In this section we will review some knowledge about βS, the space of ultrafilters over a set S, so that we can provide a thorough proof of Theorems 6.2.8 and 6.6.1 as Theorems 6.9.12 and 6.9.18 respectively. After proving Theorem 6.9.18, we give a brief discussion comparing the methods that we use to show that certain equations are partition regular with the methods used in [Ber10] and [Hin11]. As a result of this discussion (cf. Remark 6.9.13), we decide to prove a generalization of Theorem 6.9.12 as Corollary 6.9.26. For a more comprehensive study of ultrafilters and their usage in the study of semigroups the reader is referred to [HS12].

Let us recall some notation before proceeding further. We let $P(S)$ denote the collection of subsets of S and $P_f(S)$ denotes the collection of finite subsets of S. If $(S, \ )$ is a semigroup, then for $s \quad S$ and $A \quad S$ we define $sA = \{sa \mid a \quad A\}$ and $s^{-1}A = \{x \quad S \mid sx \quad A\}$.

Definition 6.9.1 (cf. Def. 6.2.7). *Let S be a set. $p \subseteq \mathcal{P}(S)$ is an **ultrafilter** if it satisfies the following properties:*

(i) $\emptyset \notin p$.

(ii) If $A \in p$ *and* $A \subseteq B$ *then* $B \in p$.

(iii) If $A, B \in p$ *then* $A \cap B \in p$.

(iv) For any $A \subseteq S$, *either* $A \in p$ *or* $A^c \in p$.

βS *denotes the space of all ultrafilters over* S.

It is often useful to think about βS as the set of finitely additive $\{0, 1\}$-valued measures on the set S. The topology of βS is generated by the basis of open sets $\{\hat{A}\}_{A \subseteq S}$, where

$$\hat{A} := \{p \in \beta S \mid A \in p\}. \tag{6.153}$$

Since $(\hat{A})^c = \widehat{A^c}$ for all $A \subseteq S$, we see that each $\hat{A}$ is also a closed set, and it is a fact that $\{\hat{A}\}_{A \subseteq S}$ also generates the topology of βS as a basis of closed sets. We note that for any $s \in S$, the collection of sets given by $e_s := \{A \subseteq S \mid s \in A\}$ is an ultrafilter over S. Let $e : S \to \beta S$ be given by $e(s) = e_s$ and observe that e is an injective map that naturally embeds S inside of βS as a dense subset. Furthermore, when we endow S with the discrete topology, which will always be the case for the rest of this section, e is a homeomorphism onto its image. An ultrafilter p is a principal ultrafilter if $p \in e(S)$, and a nonprincipal ultrafilter otherwise. Since we naturally identify each $s \in S$ with the principal ultrafilter e_s, it is common to write s in place of e_s, and we will be using this convention for the rest of this section. Theorem 6.9.2 is a universal property that can be used to characterize βS.

Theorem 6.9.2 (cf. Theorem 3.28 in [HS12]). *Let S be an infinite set with the discrete topology and let βS denote the space of ultrafilters over S. Given any compact space Y and any function $f : S \to Y$ there exists a unique continuous function $\tilde{f} : \beta S \to Y$ such that $\tilde{f}|_S = f$.*

A careful and repeated application of Theorem 6.9.2 also allows one to extend binary operations from $S \times S$ to $\beta S \times \beta S$, which is of great use when S is a semigroup.

Theorem 6.9.3 (cf. Theorem 4.1 in [HS12]). *Let S be a set and let $*$ be a binary operation defined on S. There is a unique binary operation $* : \beta S \times \beta S \to \beta S$ satisfying the following three conditions:*

(a) For every $s, t \in S$, $s * t = s * t$.

(b) For each $q \quad \beta S$*, the function* $\rho_q : \beta S \to \beta S$ *is continuous, where* $\rho_q(p) = p \quad q$*.*

(c) For each $s \quad S$*, the function* $\lambda_s : \beta S \to \beta S$ *is continuous, where* $\lambda_s(q) = s \quad q$*.*

Theorem 6.9.4 (cf. Theorems 4.4 and 4.12 in [HS12])**.** *If* $(S, \ )$ *is a semigroup and* $\ : \beta S \times \beta S \to \beta S$ *is the operation given by Theorem 6.9.3, then is an associative operation. Furthermore, for* $p, q \quad \beta S$ *the ultrafilter* $p \quad q$ *is given by*

$$A \quad p \quad q \Leftrightarrow \{s \quad S \mid s^{-1}A \quad q\} \quad p. \tag{6.154}$$

It is customary to denote the operation that is produced by Theorem 6.9.3 by so that represents an operation on $S \times S$ as well as $\beta S \times \beta S$, and we shall adopt this practice. In light of Theorem 6.9.4 We see that if $(S, \ )$ is a semigroup, then $(\beta S, \ )$ is also a semigroup. Since βS is a compact Hausdorff space (cf. Theorem 3.18 in [HS12]), and for each $q \quad \beta\mathbb{N}$ the map $\rho_q : \beta\mathbb{N} \to \beta\mathbb{N}$ given by $\rho_q(p) = p \quad q$ is continuous (cf. Theorem 6.9.3), we see that $(\beta S, \ )$ is a compact right topological semigroup.[3] We apologize to the reader for our seemingly excessive emphasis on the operation of our semigroup $(S, \ )$, but we choose to do this to avoid confusion later on when we work with rings $(R, +, \cdot)$ and are forced to consider the semigroups $(R, +)$ and $(R, \cdot)$ separately. We will now collect some facts from the theory of semigroups.

Definition 6.9.5. *Let* $(S, \ )$ *be a semigroup.*

- *We say* $e \quad S$ *is an* ***idempotent*** *if* $e \quad e = e$*. We let* $E(S, \ )$ *denote the set of idempotents of* $(S, \ )$*.*

- *We say* $L \quad S$ *is a* ***left ideal*** *if for any* $s \quad S$ *and* $\ell \quad L$ *we have* $s \quad \ell \quad L$*. We say* $R \quad S$ *is a* ***right ideal*** *if for any* $s \quad S$ *and* $r \quad R$ *we have* $r \quad s \quad R$*. In general,* $I \quad S$ *is an* ***ideal*** *if it is a left ideal and a right ideal.*

- *We call* $L \quad S$ *a* ***minimal left ideal*** *if* L *is a left ideal that does not properly contain any other left ideal. Similarly,* ***the smallest ideal of S****, if it exists, is an ideal* I *that is contained in every other ideal of* S*.*[4] *If* $(S, \ )$ *does possess a smallest ideal, then we denote it by* $K(S, \ )$ *and observe that* $K(S, \ )$ *is also a semigroup.*

---

[3] A **compact right topological semigroup** is a semigroup $(S, \ )$ that is also a compact Hausdor space for which each of the maps $\rho_s : S \to S$ given by $\rho_s(t) = t \quad s$ are continuous. Note that other sources may use $\rho_s(t) = s \quad t$ in their definition and that we base our definition o of Definition 2.1 in [HS12].

[4] Note that not every semigroup possesses a smallest ideal. In the semigroup $(\mathbb{N}, +)$ each of the sets $I_n := \{m \quad \mathbb{N} \mid m \geq n\}$ is an ideal. Since there is no set that is contained in every I_n, we see that $(\mathbb{N}, +)$ does not have a smallest ideal.

Theorem 6.9.6 (cf. Corollary 2.6 in [HS12])**.** *Let $(S, \ )$ be a compact right topological semigroup. Then every left ideal of S contains a minimal left ideal. Minimal left ideals are closed, and each minimal left ideal has an idempotent.*

Theorem 6.9.7 (cf. Theorem 1.51 in [HS12])**.** *Let $(S, \ )$ be a semigroup. If S has a minimal left ideal, then $K(S, \ )$ exists and $K(S, \ ) = \bigcup\{L \mid L$ is a minimal left ideal of $S\}$.*

Lemma 6.9.8. *Let $(S, \ )$ be a semigroup and let $\leq$ be the partial ordering on the set of idempotents of S given by $f \leq e$ if and only if $fe = ef = f$. Assume that S has a minimal left ideal that has an idempotent.*

*(i) If $e \quad S$ is an idempotent that is minimal with respect to $\leq$, then e is a member of some minimal left ideal of S. Such an idempotent is a **minimal idempotent**. It follows that $E(K(S, \ ))$ is the set of minimal idempotents of $(S, \ )$. (cf. Theorem 1.38(d) in [HS12])*

(ii) If $f \quad S$ is an idempotent then there exists a minimal idempotent e such that $e \leq f$. (cf. Theorem 1.60 in [HS12])

Definition 6.9.9. *Let $(S, \ )$ be a semigroup and let $A \quad S$. Then A is **central** if there exists an ultrafilter $p \quad E(K(\beta S, \ ))$ for which $A \quad p$.*

Theorem 6.9.10 (cf. Theorem 3.5 in [BJM17])**.** *Let $(S, \ )$ be a commutative semigroup, let $\ell \quad \mathsf{N}$, and let $A \quad \mathsf{N}$ be a central set.*

(i) There exists $b, g \quad A$ such that

$$b, b \quad g, b \quad g \quad g, \cdots, b \quad \underbrace{g \quad g \quad \cdots \quad g}_{\ell} \quad A. \tag{6.155}$$

(ii) If $(S, \ )$ is a commutative group and $c : S \to S$ is a homomorphism for which $[S : c(S)] < \infty$, then there exists $b, g \quad S$ such that

$$\{c(g)\} \quad \{c(b) \quad g^j\}_{j=-\ell}^{\ell} \quad A. \tag{6.156}$$

Lemma 6.9.11 (cf. Lemma 17.2 and Theorem 17.3 in [HS12])**.** *There exists an ultrafilter $p \quad \beta\mathsf{N}$ such that every $A \quad p$ is a central subset of $(\mathsf{N}, +)$ and a central subset of $(\mathsf{N}, \cdot)$.*

We are now ready to prove Theorem 6.2.8.

Theorem 6.9.12 (cf. Theorem 6.2.8)**.** *Let* $p \quad \beta\mathbb{N}$ *be such that every* $A \quad p$ *is a central subset of* $(\mathbb{N},+)$ *and a central subset of* $(\mathbb{N},\cdot)$. p *satisfies the following properties:*

(i) For any $A \quad p$ *and* $\ell \quad \mathbb{N}$*, there exists* $b, g \quad A$ *with* $\{bg^j\}_{j=0}^{\ell} \quad A$.

(ii) For any $A \quad p$ *and* $h, \ell \quad \mathbb{N}$*, there exists* $a, d \quad \mathbb{N}$ *for which* $\{hd\} \quad \{ha+id\}_{i=-\ell}^{\ell} \quad A$.

(iii) For any $s \quad \mathbb{N}$*, we have* $s\mathbb{N} \quad p$.

Proof. By Lemma 6.9.11 let $p \quad \beta\mathbb{N}$ be an ultrafilter such that every $A \quad p$ is a central subset of $(\mathbb{N},+)$ and a central subset of $(\mathbb{N},\cdot)$. To see that p satisfies (i), it suffices to observe that each $A \quad p$ is a central subset of $(\mathbb{N},\cdot)$, so Theorem 6.9.10(i) applies. To see that p satisfies (ii) we first note that any central subset of $(\mathbb{N},+)$ is also a central subset of $(\mathbb{Z},+)$ (cf. Theorem 2.9 in [Phu15]), so we may apply Theorem 6.9.10(ii) with the homomorphism c given by $c(x) = hx$ for all $x \quad \mathbb{Z}$ to find some $a, d \quad \mathbb{Z}$ for which $\{hd\} \quad \{ha+id\}_{i=-\ell}^{\ell} \quad A$. Since $ha, hd \quad A \quad \mathbb{N}$, we see that $a, d \quad \mathbb{N}$. To see that p satisfies (iii), let $s \quad \mathbb{N}$ be arbitrary and let us assume for the sake of contradiction that $s\mathbb{N} \ / \ p$. It follows that $(s\mathbb{N})^c \quad p$, so by (ii) let $a, d \quad \mathbb{N}$ be such that $\{sd\} \quad \{sa+id\}_{i=-1}^{1} \quad (s\mathbb{N})^c$ to obtain the desired contradiction. □

Remark 6.9.13. In [Ber10] and [Hin11] the authors also used an ultrafilter p for which every $A \quad p$ is a central subset of $(\mathbb{N},+)$ and a central subset of $(\mathbb{N},\cdot)$ in order to show that the equation $x+y = wz$ is partition regular over $\mathbb{N}$, so it is unsurprising that we have managed to use such an ultrafilter to obtain our positive results over $\mathbb{N}$ and $\mathbb{Z} \setminus \{0\}$. Unfortunately, if R is a general integral domain, then there may not exist an ultrafilter $p \quad \beta R$ for which every $A \quad p$ is a central subset of $(R,+)$ and a central subset of $(R \setminus \{0\},\cdot)$. Thankfully, we will see as a consequence of Theorem 6.9.18 that we only need to work with central subsets of $(R \setminus \{0\},\cdot)$ in order to get our desired results for a general integral domain. In particular, the ultrafilter from Theorems 6.6.1 and 6.9.18 is just a minimal idempotent in $(\beta R \setminus \{0\},\cdot)$, so it is a corollary of Lemma 6.6.2 that any central subset of $(\mathbb{Z} \setminus \{0\},\cdot)$ (and consequently of $(\mathbb{N},\cdot)$) contains a solution to the equation $x+y = wz$. For the sake of completeness, we will still examine rings R for which there exists an ultrafilter $p \quad \beta R$ such that every $A \quad p$ is a central subset of $(R,+)$ and a central subset of $(R \setminus \{0\},\cdot)$ after we prove Theorem 6.9.18.

We would also like to point out to the reader that we work with central subsets of $(R \setminus \{0\},\cdot)$ instead of central subsets of $(R,\cdot)$ because $K(\beta R,\cdot) = 0$, so $\{0\}$ is the only central subset of $(R,\cdot)$. For central subsets of $(R \setminus \{0\},\cdot)$ to be defined, we need $(R \setminus \{0\},\cdot)$ to be a semigroup, which is why we will only work with division rings and integral domains for the rest of this section. We also observe that the natural inclusion map $\iota : \beta(R \setminus \{0\}) \to (\beta R) \setminus \{0\}$ given by

$$\iota(p) = \{A \quad R \mid A \setminus \{0\} \quad p\} = p \quad \{A \quad \{0\} \mid A \quad p\} \tag{6.157}$$

is a homeomorphism. Furthermore, we see that for any $p_1, p_2 \quad \beta(R \setminus \{0\})$ we have that $\iota(p_1 \cdot p_2) = \iota(p_1) \cdot \iota(p_2)$, so ι is a semigroup isomorphism as well. Since $(\beta R) \setminus \{0\}$ and $\beta(R \setminus \{0\})$ are naturally isomorphic as compact right topological semigroups, we will write $\beta R \setminus \{0\}$ for $(\beta R) \setminus \{0\}$ without any worry for the potential confusion with $\beta(R \setminus \{0\})$.

Theorem 6.9.14 (cf. Theorem A in [BK21])**.** *Let R be an infinite integral domain and let M be a matrix with entries in R. Then the system $M\vec{x} = \vec{0}$ is partition regular over $R \setminus \{0\}$ if and only if M satisfies the columns condition (cf. Definition 6.2.2).*[5]

Lemma 6.9.15. *Let R be an infinite integral domain and let M be a matrix with entries in R for which the system $M\vec{x} = \vec{0}$ is partition regular over $R \setminus \{0\}$. Let $I_M \quad \beta R \setminus \{0\}$ denote the collection of ultrafilters p such that every $A \quad p$ there exists $\vec{x}$ with entries from A such that $M\vec{x} = \vec{0}$. I_M is an ideal of $(\beta R \setminus \{0\}, \cdot)$.*

Proof. First let us show that $I_{\mathbf{M}}$ is nonempty. To this end, let us assume for the sake of contradiction that for each $p \quad \beta R \setminus \{0\}$ there exists $A_p \quad p$ such that there is no $\vec{x}$ with entries in A_p satisfying $\mathbf{M}\vec{x} = \vec{0}$. Since $\{\widehat{A_p}\}_{p \quad \beta R \setminus \{0\}}$ is an open cover of the compact space $\beta R \setminus \{0\}$, let $\{\widehat{A_{p_i}}\}_{i=1}^r$ be a finite subcover. The desired contradiction follows from the observation that $R \setminus \{0\} = \bigcup_{i=1}^r A_{p_i}$ is a partition in which no cell yields a solution to the equation $\mathbf{M}\vec{x} = \vec{0}$. Now let us show that I_F is a left ideal. To this end, let $p \quad I_F$ and $q \quad \beta R \setminus \{0\}$ both be arbitrary. We see that for $A \quad q \cdot p$ we have

$$\{r \quad R \setminus \{0\} \mid r^{-1}A \quad p\} \quad q, \tag{6.158}$$

so let $r \quad R \setminus \{0\}$ be such that $r^{-1}A \quad p$. Since $r^{-1}A \quad p$, let $x_1, \cdots, x_k \quad r^{-1}A$ be a solution to F. Since F is homogeneous, we see that $rx_1, \cdots, rx_k \quad A$ is also a solution to F, which completes the proof that I_F is a left ideal. Now let us show that I_F is a right ideal. To this end, let $p \quad I$ and $q \quad \beta R \setminus \{0\}$ both be arbitrary. We see that for $A \quad p \cdot q$ we have

$$\{r \quad R \setminus \{0\} \mid r^{-1}A \quad q\} \quad p, \tag{6.159}$$

so let $x_1, \cdots, x_k \quad \{r \quad R \setminus \{0\} \mid r^{-1}A \quad q\}$ be a solution to F. Since $\bigcap_{i=1}^k x_i^{-1}A \quad q$, let $y \quad \bigcap_{i=1}^k x_i^{-1}A$ be arbitrary and note that $x_1y, \cdots, x_ky \quad A$ is a solution to F since F is a homogeneous system of equations. □

[5] In [BK21] the statements of the results discuss partition regularity over R, not $R \setminus \{0\}$, but the definition of partition regularity that is used in [BK21] explicitly forbids trivial solutions, which is why we may modify their statement to mirror our previous statements.

Corollary 6.9.16. *Let R be an infinite integral domain. Let $I \quad \beta R \setminus \{0\}$ denote the collection of ultrafilters p such that for every $A \quad p$ and every finite system of homogeneous linear equations F that is partition regular over $R \setminus \{0\}$, there is a solution to F contained in A. Then I is a nonempty ideal of $(\beta R \setminus \{0\}, \cdot)$.*

Proof. Let $\mathcal{M}$ denote the collection of matrices $\mathbf{M}$ with entries in R for which the system $\mathbf{M}\vec{x} = \vec{0}$ is partition regular. We see that

$$K(\beta R \setminus \{0\}, \cdot) \quad \bigcap_{\mathbf{M} \quad \mathcal{M}} I_{\mathbf{M}} = I. \tag{6.160}$$

We see that I is an ideal since it is an intersection of ideals, and I is nonempty since it contains $K(\beta R \setminus \{0\}, \cdot)$. □

Lemma 6.9.17. *Let R be an infinite integral domain. Let $H \quad \beta R \setminus \{0\}$ denote the collection of ultrafilters p such that for every $r \quad R \setminus \{0\}$ we have $rR \quad p$. Then H is an ideal of $(\beta R \setminus \{0\}, \cdot)$.*

Proof. Let $p \quad H, q \quad \beta R \setminus \{0\}$, and $r \quad R \setminus \{0\}$ all be arbitrary. Let us first show that H is a left ideal. We note that for any $s \quad R$, we have $s^{-1}(rR) \quad rR$. Since $rR \quad p$, we see that

$$R \quad \{s \quad R \setminus \{0\} \mid s^{-1}(rR) \quad p\} \quad \{s \quad R \setminus \{0\} \mid s^{-1}(rR) \quad p\} \quad q. \tag{6.161}$$

It follows that $rR \quad q \cdot p$, so $q \cdot p \quad H$ and H is indeed a left ideal. Now let us show that H is a right ideal. We note that for any $s \quad rR$ we have $s^{-1}(rR) \quad R$. Since $R \quad q$, we see that

$$rR \quad \{s \quad R \setminus \{0\} \mid s^{-1}(rR) \quad q\} \quad \{s \quad R \setminus \{0\} \mid s^{-1}(rR) \quad q\} \quad p. \tag{6.162}$$

It follows that $rR \quad p \cdot q$, so $p \cdot q \quad H$. □

Theorem 6.9.18 (cf. Theorem 6.6.1). *Let R be an infinite integral domain. If p $E(K(\beta R \setminus \{0\}, \cdot))$, then p satisfies the following properties:*

(i) For any $A \quad p$ and $\ell \quad \mathbb{N}$, there exists $b, g \quad A$ with $\{bg^j\}_{j=0}^{\ell} \quad A$.

(ii) For any $A \quad p$ and any finite system of homogeneous linear equations F that is partition regular over $R \setminus \{0\}$, there is a solution to F contained in A.

(iii) For every $r \quad R \setminus \{0\}$, we have $rR \quad p$.

(iv) For any $A \quad p, \ell \quad \mathbb{N}$*, and* $h, j_1, j_2, \cdots, j_\ell \quad R \setminus \{0\}$*, there exists* $a, d \quad R$ *for which* $\{hd, ha\} \quad \{ha + j_i d\}_{i=0}^{\ell} \quad A$*.*

Proof. Since p is a minimal idempotent in $(\beta R \setminus \{0\}, 0)$ we see that each $A \quad p$ is a central subset of $(R \setminus \{0\}, \cdot)$, so Theorem 6.9.10(i) shows us that p satisfies condition (i). Since I and H (cf. Corollary 6.9.16 and Lemma 6.9.17 respectively) are ideals of $(\beta R \setminus \{0\}, \cdot)$, $I \cap H$ is also an ideal. Hence $K(\beta R \setminus \{0\}, \cdot) \quad I \cap H$. Since $p \quad I$, we see that condition (ii) is satisfied. Since $p \quad H$, we see that condition (iii) is satisfied. The fact that p satisfies (iv) when R is an integral domain is a corollary of the fact that p satisfies (ii). We give a proof of this implication for the sake of completeness since property (iv) is used in the earlier sections of this paper.

To see that p satisfies condition (iv), we first consider the system of equations

$$\begin{array}{lllllll} hx_3 & - & hx_2 & - & j_1 x_1 & = & 0 \\ hx_4 & - & hx_2 & - & j_2 x_1 & = & 0 \\ & & & \vdots & & & \\ hx_{\ell+2} & - & hx_2 & - & j_\ell x_1 & = & 0 \\ hx_{\ell+3} & - & x_2 & - & hx_{\ell+5} & = & 0 \\ hx_{\ell+4} & - & hx_2 & - & x_1 & = & 0 \end{array} \tag{6.163}$$

Let us assume that $\{x_i\}_{i=1}^{\ell+5} \quad R \setminus \{0\}$ is a solution to the system of equations in (6.163). Since $x_1 = h(x_{\ell+5} - x_2)$ and $x_2 = h(x_{\ell+3} - x_{\ell+4})$, we may write $x_1 = hd$ and $x_2 = ha$ for some $a, d \quad R \setminus \{0\}$. It follows that $x_i = ha + j_i d$ for $3 \leq i \leq \ell + 2$, so it suffices to show that each $A \quad p$ contains a solution to the system of equations in (6.163). To this end, let $\mathbf{M} \quad M_{\ell+3,\ell+5}(R)$ be the matrix such that the equation $\mathbf{M}\vec{x} = \vec{0}$ represents the system of equations in (6.163), and we will proceed to show that $\mathbf{M}$ satisfies the columns condition. Let $\{\vec{c}_i\}_{i=1}^{\ell+5}$ denote the columns of $\mathbf{M}$, with $\vec{c}_i$ representing the column corresponding to x_i. We see that

$$\vec{c}_1 = \begin{pmatrix} -j_1 \\ -j_2 \\ \vdots \\ -j_\ell \\ 0 \\ -1 \end{pmatrix}, \vec{c}_2 = \begin{pmatrix} -h \\ -h \\ \vdots \\ -h \\ -1 \\ -h \end{pmatrix}, \vec{c}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for } 3 \leq i \leq \ell + 4, \text{ and } \vec{c}_{\ell+5} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -h \\ 0 \end{pmatrix} \tag{6.164}$$

with the h in $\vec{c}_i$ occuring in row $i - 2$ for $3 \leq i \leq \ell + 4$. Consider the partition of column indices $\{C_1, C_2, C_3\}$ given by $C_1 = \{\ell+3, \ell+5\}, C_2 = \{2, 3, \cdots, \ell+2, \ell+4\}$, and $C_3 = \{1\}$. We see that

$$
\begin{aligned}
\vec{s}_1 &= \vec{c}_{\ell+3} + \vec{c}_{\ell+5} = \vec{0}, \\
\vec{s}_2 &= \vec{c}_{\ell+4} + \sum_{i=2}^{\ell+2} \vec{c}_i = \tfrac{1}{h}\vec{c}_{\ell+5}, \text{ and} \\
\vec{s}_3 &= \vec{c}_1 = \left(\sum_{i=3}^{\ell+2} -\tfrac{j_{i-2}}{h}\vec{c}_i\right) - \tfrac{1}{h}\vec{c}_{\ell+4},
\end{aligned}
\tag{6.165}
$$

so $\mathbf{M}$ does indeed satisfy the columns condition. □

To conclude this section we will show that an integral domain R possesses an ultrafilter p βR such that every A p is a central subset of $(R, +)$ and a central subset of $(R, \cdot)$ if and only if R is homomorphically finite.

Definition 6.9.19 (cf. Def. 4.38 in [HS12]). *Let $(S, \ )$ be a semigroup and let A S. A is **syndetic** if and only if there exists some G $\mathcal{P}_f(S)$ such that $S = \bigcup_{t \ G} t^{-1}A$.*

We observe that if $(S, \ )$ is a group and H S is a subgroup, then H is a syndetic subset of S if and only if $[S : H] < \infty$.

Definition 6.9.20. *A ring R is a **right (left) homomorphically finite** if for every r $R \setminus \{0\}$ the right (left) ideal rR (Rr) is a finite index subgroup of $(R, +)$. R is a **homomorphically finite** if for every r $R \setminus \{0\}$ the two-sided ideal RrR is a finite index subgroup of $(R, +)$.*

Theorem 6.9.21 (cf. Corollary 1.3.3 in [Coh95]). *Let R be a ring with no zero divisors*[6] *such that*

$$
aR \cap bR = \{0\} \quad a, b \quad R \setminus \{0\}. \tag{6.166}
$$

Then the localization of R at $R \setminus \{0\}$ is a division ring D and the natural homomorphism $\lambda : R \to D$ is an embedding.

Theorem 6.9.22 (cf. Theorem 5.8 in [HS12]). *Let $(S, \ )$ be a semigroup, let p be an idempotent in βS, and let A p. There is a sequence $(x_n)_{n=1}^{\infty}$ in S such that $\{\prod_{n \ F} x_n\}_{F \ \mathcal{P}_f(\mathbb{N})}$ A.*

Theorem 6.9.23. *Let $(G, \ )$ be a group and let H be a finite index subgroup of G. If p βG is an idempotent, then H p.*

[6] The reader is warned that in [Coh95] an integral domain is a not necessarily commutative ring with no zero divisors. Similarly, in [Coh95] a field refers to a not necessarily commutative division ring.

Proof. Let $M = [G : H]$ and let us assume for the sake of contradiction that $H^c \quad p$. By Theorem 6.9.22 let $(x_n)_{n=1}^{\infty}$ be a sequence in G such that $\{\prod_{n \quad F} x_n\}_{F \quad P_f(\mathbb{N})} \quad H^c$. Since H^c is a disjoint union of $M - 1$ cosets of H, let $1 \leq j < k \leq M$ be such that $(\prod_{i=1}^{j} x_i)H = (\prod_{i=1}^{k} x_i)H$. It follows that $(\prod_{i=j+1}^{k} x_i)H = H$, hence $\prod_{i=j+1}^{k} x_i \quad H$, which yields the desired contradiction. □

Lemma 6.9.24. *If R is an infinite right (respectively left) homomorphically finite ring that has no zero divisors and $p \quad E(K(\beta R, +))$, then for any $r \quad R \setminus \{0\}$ we have $r \cdot p \quad E(K(\beta R, +))$ (respectively $p \cdot r \quad E(K(\beta R, +))$).*

Proof. We only prove the desired result for $r \cdot p$ since the proof of the result for $p \cdot r$ is similar. Firstly, we would like to show that R is a subring of a division ring D, so it suffices to show that R satisfies the conditions of Theorem 6.9.21. Let $r, s \quad R \setminus \{0\}$ be arbitrary and note that within the group $(R, +)$ we have $[R : rR \cap sR] \leq [R : rR][R : sR] < \infty$. Since R is infinite and has no zero divisors, we see that $|rR \cap sR| = \infty$, hence R embeds in some division ring D.

Since R is a ring, we see that for any $r \quad R$ the map $\ell_r : R \to R$ given by $\ell_r(s) = rs$ is a endomorphism of the group $(R, +)$, hence its unique continuous extension $\tilde{\ell_r} : \beta R \to \beta R$ is also an additive endomorphism (cf. Lemma 2.14 in [HS12]), so $r \cdot p = \ell_r(p)$ is an additive idempotent. It only remains to show that $r \cdot p$ is minimal. To this end, by lemma 6.9.8(ii) let $q \quad \beta R$ be a minimal idempotent for which $q \leq r \cdot p$. We note that $\tilde{\ell_{r^{-1}}} : \beta D \to \beta D$ is also an additive endomorphism, hence $r^{-1} \cdot q \leq r^{-1} \cdot r \cdot p = p$. By lemma 6.9.23 we see that $rR \quad q$, so $R \quad r^{-1} \cdot q$, hence $r^{-1} \cdot q \quad \beta R \quad \beta D$. Since $p \quad \beta R$ is minimal and $r^{-1} \cdot q \leq p$, we see that $r^{-1} \cdot q = p$, hence $q = r \cdot p$. □

Theorem 6.9.25. *If R is an infinite right homomorphically finite ring that has no zero divisors, then there exists an ultrafilter $p \quad E(K(\beta(R) \setminus \{0\}, \cdot)) \cap \overline{E(K(\beta R, +))}$.*

Proof. Using Lemma 6.9.24 and the continuity of right multiplication we see that

$$(R \setminus \{0\}) \cdot E(K(\beta R, +)) \quad E(K(\beta R, +)) \quad \overline{(R \setminus \{0\})} \cdot E(K(\beta R, +)) \quad \overline{E(K(\beta R, +))} \tag{6.167}$$

$$(\beta(R) \setminus \{0\}) \cdot E(K(\beta R, +)) \quad \overline{E(K(\beta R, +))}. \tag{6.168}$$

Since $(\beta(R) \setminus \{0\}) \cdot E(K(\beta R, +))$ is a left ideal of $(\beta R \setminus \{0\}, \cdot)$, it contains a minimal idempotent, which is the desired p. □

Corollary 6.9.26. *Let R be an infinite right homomorphically finite ring that has no zero divisors. There exists an ultrafilter $p \quad \beta R$ such that every $A \quad p$ is a central subset of $(R, +)$ and a central subset of $(R \setminus \{0\}, \cdot)$.*

Proof. Using Theorem 6.9.25 let us pick some $p \quad E(K(\beta R \setminus \{0\}, \cdot)) \cap \overline{E(K(\beta R, +))}$ and let $A \quad p$ be arbitrary. Since $p \quad E(K(\beta R \setminus \{0\}, \cdot)) = E(K(\beta(R \setminus \{0\}), \cdot))$, we see that A is a central subset of $(R \setminus \{0\}, \cdot)$. Since $p \quad \overline{E(K(\beta R, +))}$ and $\hat{A}$ is an open neighborhood of p, pick some $q \quad E(K(\beta R, +)) \cap \hat{A}$. Since $A \quad q$ and $q \quad E(K(\beta R, +))$, we see that A is a central subset of $(R, +)$. □

Remark 6.9.27. To see why we need to assume that R is a right homomorphically finite ring let us consider the integral domain $\mathsf{Q}[x]$. It is clear that $x\mathsf{Q}[x]$ is not a finite index subgroup of $(\mathsf{Q}[x], +)$, and we will see in Theorem 6.9.28 that $x\hat{\mathsf{Q}}[x] \quad \overline{K(\beta\mathsf{Q}[x], \cdot)}$ while $x\hat{\mathsf{Q}}[x] \cap \overline{K(\beta\mathsf{Q}[x], +)} = \quad$. It follows that $x\hat{\mathsf{Q}}[x] \quad \overline{E(K(\beta\mathsf{Q}[x], \cdot))}$ while $x\hat{\mathsf{Q}}[x] \cap \overline{E(K(\beta\mathsf{Q}[x], +))} = \quad$, so $x\mathsf{Q}[x]$ intersects every central subset of $(\mathsf{Q}[x] \setminus \{0\}, \cdot)$ even though it is not a central subset of $(\mathsf{Q}[x], +)$. Furthermore, in the proof of lemma 6.9.25 we used the continuity of right multiplication in $\beta(R \setminus \{0\})$ which is why we had to assume that the ring R was a right homomorphically finite ring. The same proof yields an analogous result for left homomorphically finite rings if you work with the extension of $\cdot$ from $R \setminus \{0\}$ to $\beta(R \setminus \{0\})$ that makes left multiplication continuous. Note that the minimal idempotents of $\beta(R \setminus \{0\}, \cdot)$ may change depending on which extension of $\cdot$ from $R \setminus \{0\}$ to $\beta(R \setminus \{0\})$ you use (cf. Theorem 13.40.2 in [HS12]).

Theorem 6.9.28. *Let R be an infinite integral domain that is not homomorphically finite and let $r_0 \quad R \setminus \{0\}$ be such that $[R : r_0 R] = \infty$. We have*

$$\widehat{r_0 R} \quad \overline{K(\beta R \setminus \{0\}, \cdot)} \text{ and } \widehat{r_0 R} \cap \overline{K(\beta R, +)} = \quad . \tag{6.169}$$

In particular, we have

$$\overline{K(\beta R \setminus \{0\}, \cdot)} \cap \overline{K(\beta R, +)} = \quad . \tag{6.170}$$

Proof. Letting H be as in Lemma 6.9.17 and observe that

$$H = \bigcap_{r \quad R \setminus \{0\}} \widehat{rR}, \tag{6.171}$$

so H is a closed. Since H is an ideal by Lemma 6.9.17, we see that $K(\beta R \setminus \{0\}, \cdot) \quad H$, hence

$$\overline{K(\beta R \setminus \{0\}, \cdot)} \quad \overline{H} = H \quad \widehat{r_0 R}. \tag{6.172}$$

Now let us assume for the sake of contradiction that there exists some $p \in \widehat{r_0R} \cap \overline{K(\beta R, +)}$. Since $p \in \overline{K(\beta R, +)}$ and $\widehat{r_0R}$ is an open neighborhood of p, pick some $q \in \widehat{r_0R} \cap K(\beta R, +)$. Theorem 4.39 of [HS12] states that $\{r \in R \mid -r + r_0R \in q\}$ is a syndetic subset of $(R, +)$. Noting that for each $r \in R$ we have $-r + r_0R = r_0R$ if $r \in r_0R$ and $(-r + r_0R) \cap r_0R = \emptyset$ if $r \notin r_0R$, we see that $\{r \in R \mid -r + r_0R \in q\} = r_0R$. Since $[R : r_0R] = \infty$, we see that r_0R is not a syndetic subset of $(R, +)$, which yields the desired contradiction. □

Corollary 6.9.29. *Let R be an infinite integral domain that is not homomorphically finite. There does not exist an ultrafilter $p \in \beta R$ such that every $A \in p$ is a central subset of $(R, +)$ and a central subset of $(R \setminus \{0\}, \cdot)$.*

Proof. Let us assume for the sake of contradiction that such an ultrafilter $p \in \beta R$ did exist. Let $A \in p$ be arbitrary. Since A is a central subset of $(R \setminus \{0\}, \cdot)$ let $q' \in E(K(\beta R \setminus \{0\}, \cdot))$ be such that $A \in q'$. Since A is a central subset of $(R, +)$, let $q \in E(K(\beta R, +))$ be such that $A \in q$. We see that $\hat{A}$ is an open neighborhood of p that contains q and q', hence

$$p \in \overline{E(K(\beta R \setminus \{0\}, \cdot))} \cap \overline{E(K(\beta R, +))}, \tag{6.173}$$

which contradicts Theorem 6.9.28. □

BIBLIOGRAPHY

[BB86] D. Berend and V. Bergelson. "Mixing sequences in Hilbert spaces". In: *Proc. Amer. Math. Soc.* 98.2 (1986), pp. 239–246. ISSN: 0002-9939.

[BE74] J. Blum and B. Eisenberg. "Generalized summing sequences and the mean ergodic theorem". In: *Proc. Amer. Math. Soc.* 42 (1974), pp. 423–429. ISSN: 0002-9939.

[Ber00] V. Bergelson. "The multifarious Poincaré recurrence theorem". In: *Descriptive set theory and dynamical systems (Marseille-Luminy, 1996)*. Vol. 277. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2000, pp. 31–57.

[Ber10] V. Bergelson. "Ultrafilters, IP sets, dynamics, and combinatorial number theory". In: *Ultrafilters across mathematics*. Vol. 530. Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 23–47.

[Ber+14] V. Bergelson, A. del Junco, M. Lemańczyk, and J. Rosenblatt. "Rigidity and non-recurrence along sequences". In: *Ergodic Theory Dynam. Systems* 34.5 (2014), pp. 1464–1502. ISSN: 0143-3857.

[Ber86] V. Bergelson. "A density statement generalizing Schur's theorem". In: *J. Combin. Theory Ser. A* 43.2 (1986), pp. 338–343. ISSN: 0097-3165.

[Ber87] V. Bergelson. "Weakly mixing PET". In: *Ergodic Theory Dynam. Systems* 7.3 (1987), pp. 337–349. ISSN: 0143-3857.

[Ber96] V. Bergelson. "Ergodic Ramsey theory—an update". In: *Ergodic theory of $\mathbb{Z}^d$ actions (Warwick, 1993–1994)*. Vol. 228. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1996, pp. 1–61.

[BF45] H. Bohr and E. Følner. "On some types of functional spaces. A contribution to the theory of almost periodic functions". In: *Acta Math.* 76 (1945), pp. 31–155. ISSN: 0001-5962.

[BH96] V. Bergelson and I. J. Håland. "Sets of recurrence and generalized polynomials". In: *Convergence in ergodic theory and probability (Columbus, OH, 1993)*. Vol. 5. Ohio State Univ. Math. Res. Inst. Publ. de Gruyter, Berlin, 1996, pp. 91–110.

[BHK09] V. Bergelson and I. J. Håland Knutson. "Weak mixing implies weak mixing of higher orders along tempered functions". In: *Ergodic Theory Dynam. Systems* 29.5 (2009), pp. 1375–1416. ISSN: 0143-3857.

[Bir31] G. D. Birkhoff. "Proof of the ergodic theorem". In: *Proc. Natl. Acad. Sci. USA* 17 (1931), pp. 656–660.

[BJM17] V. Bergelson, J. H. Johnson Jr., and J. Moreira. "New polynomial and multidimensional extensions of classical partition results". In: *J. Combin. Theory Ser. A* 147 (2017), pp. 119–154. ISSN: 0097-3165.

[BK21] J. Byszewski and E. Krawczyk. "Rado's theorem for rings and modules". In: *J. Combin. Theory Ser. A* 180 (2021), pp. 105402, 28. ISSN: 0097-3165.

[BL02] V. Bergelson and A. Leibman. "A nilpotent Roth theorem". In: *Invent. Math.* 147.2 (2002), pp. 429–470. ISSN: 0020-9910.

[BL08] V. Bergelson and E. Lesigne. "Van der Corput sets in Z^d". In: *Colloq. Math.* 110.1 (2008), pp. 1–49. ISSN: 0010-1354.

[BL96] V. Bergelson and A. Leibman. "Polynomial extensions of van der Waerden's and Szemerédi's theorems". In: *J. Amer. Math. Soc.* 9.3 (1996), pp. 725–753. ISSN: 0894-0347.

[BLM21] J. M. Barrett, M. Lupini, and J. Moreira. "On Rado conditions for nonlinear Diophantine equations". In: *European J. Combin.* 94 (2021), pp. 103277, 20. ISSN: 0195-6698.

[BM10] V. Bergelson and R. McCutcheon. "Idempotent ultrafilters, multiple weak mixing and Szemerédi's theorem for generalized polynomials". In: *J. Anal. Math.* 111 (2010), pp. 77–130. ISSN: 0021-7670.

[BM16] V. Bergelson and J. Moreira. "Van der Corput's difference theorem: some modern developments". In: *Indag. Math. (N.S.)* 27.2 (2016), pp. 437–479. ISSN: 0019-3577.

[Bou87] J. Bourgain. "Ruzsa's problem on sets of recurrence". In: *Israel J. Math.* 59.2 (1987), pp. 150–166. ISSN: 0021-2172.

[Bou89] J. Bourgain. "Pointwise ergodic theorems for arithmetic sets". In: *Inst. Hautes Études Sci. Publ. Math.* 69 (1989). With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein, pp. 5–45. ISSN: 0073-8301.

[Bra28] R. Brauer. "Untersuchungen über die arithmetischen Eigenschaften von Gruppen linearer Substitutionen". In: *Math. Z.* 28.1 (1928), pp. 677–696. ISSN: 0025-5874.

[CGS12] P. Csikvári, K. Gyarmati, and A. Sárközy. "Density and Ramsey type results on algebraic equations with restricted solution sets". In: *Combinatorica* 32.4 (2012), pp. 425–449. ISSN: 0209-9683.

[Cha62] R. V. Chacon. "An ergodic theorem for operators satisfying norm conditions". In: *J. Math. Mech.* 11 (1962), pp. 165–172.

[Coh13] D. L. Cohn. *Measure theory*. Second. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2013, pp. xxi+457. ISBN: 978-1-4614-6955-1; 978-1-4614-6956-8.

[Coh95] P. M. Cohn. *Skew fields*. Vol. 57. Encyclopedia of Mathematics and its Applications. Theory of general division rings. Cambridge University Press, Cambridge, 1995, pp. xvi+500. ISBN: 0-521-43217-0.

[Cor31] J. G. van der Corput. "Diophantische Ungleichungen. I. Zur Gleichverteilung Modulo Eins". In: *Acta Math.* 56.1 (1931), pp. 373–456. ISSN: 0001-5962.

[DMF74] H. Daboussi and M. Mendès France. "Spectrum, almost-periodicity and equidistribution modulo 1". In: *Studia Sci. Math. Hungar.* 9 (1974), 173–180 (1975). ISSN: 0081-6906.

[DNLB18] M. Di Nasso and L. Luperi Baglini. "Ramsey properties of nonlinear Diophantine equations". In: *Adv. Math.* 324 (2018), pp. 84–117. ISSN: 0001-8708.

[Eis+15] T. Eisner, B. Farkas, M. Haase, and R. Nagel. *Operator theoretic aspects of ergodic theory*. Vol. 272. Graduate Texts in Mathematics. Springer, Cham, 2015, pp. xviii+628. ISBN: 978-3-319-16897-5; 978-3-319-16898-2.

[ET36] P. Erdös and P. Turán. "On Some Sequences of Integers". In: *J. London Math. Soc.* 11.4 (1936), pp. 261–264. ISSN: 0024-6107.

[ET57] P. Erdös and S. J. Taylor. "On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences". In: *Proc. London Math. Soc. (3)* 7 (1957), pp. 598–615. ISSN: 0024-6115.

[EW11] M. Einsiedler and T. Ward. *Ergodic theory with a view towards number theory*. Vol. 259. Graduate Texts in Mathematics. Springer-Verlag London, Ltd., London, 2011, pp. xviii+481. ISBN: 978-0-85729-020-5.

[Far21] S. Farhangi. "Pointwise ergodic theorems for higher levels of mixing". In: *Studia Math.* 261.3 (2021), pp. 329–344. ISSN: 0039-3223.

[FH] N. Frantzikinakis and B. Host. "Multiple Recurrence and Convergence without Commutativity". In: *arXiv:2111.01518v2 [math.DS]* ().

[FK78] H. Furstenberg and Y. Katznelson. “An ergodic Szemerédi theorem for commuting transformations”. In: *J. Analyse Math.* 34 (1978), 275–291 (1979). ISSN: 0021-7670.

[FLW06] N. Frantzikinakis, E. Lesigne, and M. Wierdl. “Sets of k-recurrence but not $(k+1)$-recurrence”. In: *Ann. Inst. Fourier (Grenoble)* 56.4 (2006), pp. 839–849. ISSN: 0373-0956.

[Fra22] N. Frantzikinakis. “Furstenberg systems of Hardy field sequences and applications”. In: *J. D’Analyse Math.* (2022 (To Appear)).

[Fur77] H. Furstenberg. “Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions”. In: *J. Analyse Math.* 31 (1977), pp. 204–256. ISSN: 0021-7670.

[Fur81] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory*. M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981, pp. xi+203. ISBN: 0-691-08269-3.

[Gla03] E. Glasner. *Ergodic theory via joinings*. Vol. 101. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xii+384. ISBN: 0-8218-3372-3.

[GRS13] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey theory*. Wiley Series in Discrete Mathematics and Optimization. Paperback edition of the second (1990) edition [MR1044995]. John Wiley & Sons Inc., Hoboken, NJ, 2013, pp. xiv+196. ISBN: 978-1-118-79966-6.

[Gru33] W. Grunwald. “Ein Allgemeiner Existenzsatz für Algebraische Zahlkörper”. In: *Journal für die reine und angewandte Mathematik* 169 (1933), pp. 103–107.

[Hin11] N. Hindman. “Monochromatic sums equal to products in $\mathbb{N}$”. In: *Integers* 11.4 (2011), pp. 431–439. ISSN: 1867-0652.

[HLS14] A. M. Hyde, P. D. Lee, and B. K. Spearman. “Polynomials $(x^3-n)(x^2+3)$ solvable modulo any integer”. In: *Amer. Math. Monthly* 121.4 (2014), pp. 355–358. ISSN: 0002-9890.

[HS12] N. Hindman and D. Strauss. *Algebra in the Stone-Čech compactification: Theory and applications*. Second revised and extended. De Gruyter Textbook. Walter de Gruyter & Co., Berlin, 2012, pp. xviii+591. ISBN: 978-3-11-025623-9.

[HST78] S. Hasegawa, R. Satō, and S. Tsurumi. “Vector valued ergodic theorems for a one-parameter semigroup of linear operators”. In: *Tohoku Math. J. (2)* 30.1 (1978), pp. 95–106. ISSN: 0040-8735.

[Jac56] K. Jacobs. “Ergodentheorie und fastperiodische Funktionen auf Halbgruppen”. In: *Math. Z.* 64 (1956), pp. 298–338. ISSN: 0025-5874.

[Jan97] S. Janson. *Gaussian Hilbert spaces*. Vol. 129. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1997, pp. x+340. ISBN: 0-521-56128-0.

[Kir72] R. B. Kirk. “Sets which split families of measurable sets”. In: *Amer. Math. Monthly* 79 (1972), pp. 884–886. ISSN: 0002-9890.

[KMF78] T. Kamae and M. Mendès France. “Van der Corput’s difference theorem”. In: *Israel J. Math.* 31.3-4 (1978), pp. 335–342. ISSN: 0021-2172.

[KN32] O. B. Koopman and J. v. Neumann. “Dynamical Systems of Continuous Spectra”. In: *Proc. Natl. Acad. Sci.* 18.3 (1932), pp. 255–263.

[KN74] L. Kuipers and H. Niederreiter. *Uniform distribution of sequences*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974, pp. xiv+390.

[Kre85] U. Krengel. *Ergodic theorems*. Vol. 6. De Gruyter Studies in Mathematics. With a supplement by Antoine Brunel. Walter de Gruyter & Co., Berlin, 1985, pp. viii+357. ISBN: 3-11-008478-3.

[Lan02] S. Lang. *Algebra*. third. Vol. 211. Graduate Texts in Mathematics. Springer-Verlag, New York, 2002, pp. xvi+914. ISBN: 0-387-95385-X.

[Lan94] S. Lang. *Algebraic number theory*. Second. Vol. 110. Graduate Texts in Mathematics. Springer-Verlag, New York, 1994, pp. xiv+357. ISBN: 0-387-94225-4.

[LG61] K. de Leeuw and I. Glicksberg. “Applications of almost periodic compactifications”. In: *Acta Math.* 105 (1961), pp. 63–97. ISSN: 0001-5962.

[LR20] I. Leader and P. Russell. “Inhomogeneous Partition Regularity”. In: *The Electronic Journal Of Combinatorics* 27 (2020), pp. 2–57.

[MRR19] J. Moreira, F. K. Richter, and D. Robertson. “A proof of a sumset conjecture of Erdős”. In: *Ann. of Math. (2)* 189.2 (2019), pp. 605–652. ISSN: 0003-486X.

[Neu32] J. von Neumann. “Proof of the quasi-ergodic hypothesis”. In: *Proc. Natl. Acad. Sci. U.S.A.* 18 (1932), pp. 70–82.

[NRS12] M. Ninčević, B. Rabar, and S. Slijepčević. “Ergodic characterization of van der Corput sets”. In: *Arch. Math. (Basel)* 98.4 (2012), pp. 355–360. ISSN: 0003-889X.

[Par67] K. R. Parthasarathy. *Probability measures on metric spaces*. Probability and Mathematical Statistics, No. 3. Academic Press, Inc., New York-London, 1967, pp. xi+276.

[Phu15] D. Phulara. “A generalized central sets theorem and applications”. In: *Topology Appl.* 196.part A (2015), pp. 92–105. ISSN: 0166-8641.

[Poi90] H. Poincaré. “Sur le problème des trois corps et les équations de la Dynamique”. In: *Acta Math.* 13 (1890), pp. 1–270.

[R5́8] A. Rényi. “On mixing sequences of sets”. In: *Acta Math. Acad. Sci. Hungar.* 9 (1958), pp. 215–228. ISSN: 0001-5954.

[Rad33] R. Rado. “Studien zur Kombinatorik”. In: *Math. Z.* 36.1 (1933), pp. 424–470. ISSN: 0025-5874.

[Rot52] K. Roth. “Sur quelques ensembles d’entiers”. In: *C. R. Acad. Sci. Paris* 234 (1952), pp. 388–390. ISSN: 0001-4036.

[Rot53] K. F. Roth. “On certain sets of integers”. In: *J. London Math. Soc.* 28 (1953), pp. 104–109. ISSN: 0024-6107.

[Roy88] H. L. Royden. *Real analysis*. Third. Macmillan Publishing Company, New York, 1988, pp. xx+444. ISBN: 0-02-404151-3.

[Ruz82] I. Ruzsa. “Ensembles intersectifs”. In: *Séminaire de Théorie des Nombres de Bordeaux* (1982-83).

[Ruz84] I. Z. Ruzsa. “Connections between the uniform distribution of a sequence and its differences”. In: *Topics in classical number theory, Vol. I, II (Budapest, 1981)*. Vol. 34. Colloq. Math. Soc. János Bolyai. North-Holland, Amsterdam, 1984, pp. 1419–1443.

[Sár78] A. Sárkőzy. “On difference sets of sequences of integers. I”. In: *Acta Math. Acad. Sci. Hungar.* 31.1-2 (1978), pp. 125–149. ISSN: 0001-5954.

[Sch16] I. Schur. “Uber die Kongruenz $x^m + y^m = z^m \pmod{p}$”. In: *Jahresber. Dtsch. Math* 25 (1916), pp. 114–117.

[Sze69] E. Szemerédi. “On sets of integers containing no four elements in arithmetic progression”. In: *Acta Math. Acad. Sci. Hungar.* 20 (1969), pp. 89–104. ISSN: 0001-5954.

[Sze75] E. Szemerédi. “On sets of integers containing no k elements in arithmetic progression”. In: *Acta Arith.* 27 (1975), pp. 199–245. ISSN: 0065-1036.

[Tse16] A. Tserunyan. “A Ramsey theorem on semigroups and a general van der Corput lemma”. In: *J. Symb. Log.* 81.2 (2016), pp. 718–741. ISSN: 0022-4812.

[V8́0] W. Y. Vélez. “On normal binomials”. In: *Acta Arith.* 36.2 (1980), pp. 113–124. ISSN: 0065-1036.

[Wae27] B. van der Waerden. "Beweis einer baudetschen Vermutung". In: *Nieuw Arch. Wiskd* 15 (1927), pp. 212–216.

[Wal75] P. Walters. *Ergodic theory—introductory lectures*. Lecture Notes in Mathematics, Vol. 458. Springer-Verlag, Berlin-New York, 1975, pp. vi+198.

[Wan48] S. Wang. "A counter-example to Grunwald's theorem". In: *Ann. of Math. (2)* 49 (1948), pp. 1008–1009. ISSN: 0003-486X.

[Wan50] S. Wang. "On Grunwald's theorem". In: *Ann. of Math. (2)* 51 (1950), pp. 471–484. ISSN: 0003-486X.

[Wey14] H. Weyl. "Über ein Problem aus dem Gebiete der diophantischen Approximationen". In: *Nachr. Ges. Wiss. Göttingen* Math. Phys. K1. (1914), pp. 234–244.

[Wey16] H. Weyl. "Über die Gleichverteilung von Zahlen mod. Eins". In: *Math. Ann.* 77 (1916), pp. 313–352.

[WW41] N. Wiener and A. Wintner. "Harmonic analysis and ergodic theory". In: *Amer. J. Math.* 63 (1941), pp. 415–426. ISSN: 0002-9327.

[Yos95] K. Yosida. *Functional analysis*. Classics in Mathematics. Reprint of the sixth (1980) edition. Springer-Verlag, Berlin, 1995, pp. xii+501. ISBN: 3-540-58654-7.