Some Collinearities of Cevian Triangles

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Let P be an arbitrary point in the plance of $\triangle ABC$, and let P' be the complement of P with respect to $\triangle ABC$. Without loss of generality (WLOG) let P_A be the intersection of lines AP and BC. WLOG let A_1 be the intersection of lines P_BP_C and BC, let A_2 be the intersection of line BC and the line through P parallel to P_BP_C, let A_3 be the intersection of line BC and the line through P' parallel to P_BP_C, let A_4 be the complement of P_A with respect to $\triangle ABC$, and let M_A be the midpoint of BC. In this paper we will prove the 3 results below.

(1) A_2, B_2, and C_2 are collinear. Furthermore, A_2B_2 is parallel to A_1B_1.

(2) A_3, B_3, and C_3 are collinear.

(3) AA_4, BB_4, and CC_4 concur at a point Q, which is also the isotomic conjugate of P with respect to $\triangle ABC$.

First, we will prove (1). We can see that WLOG $\triangle APB_2 \sim \triangle AP_AB_1$ and $\triangle APC_2 \sim \triangle AP_AC_1$, so $\frac{AB_2}{AB_1} = \frac{AP}{AP_A} = \frac{AC_2}{AC_1} \to B_1C_1 \parallel B_2C_2$. Similar reasoning will show that $A_1B_1 \parallel A_2B_2$, and using the well-known fact that A_1, B_1, and C_1 are collinear, we can see that A_2, B_2, and C_2 are collinear and make a line parallel to A_1B_1 as desired. $\blacksquare$

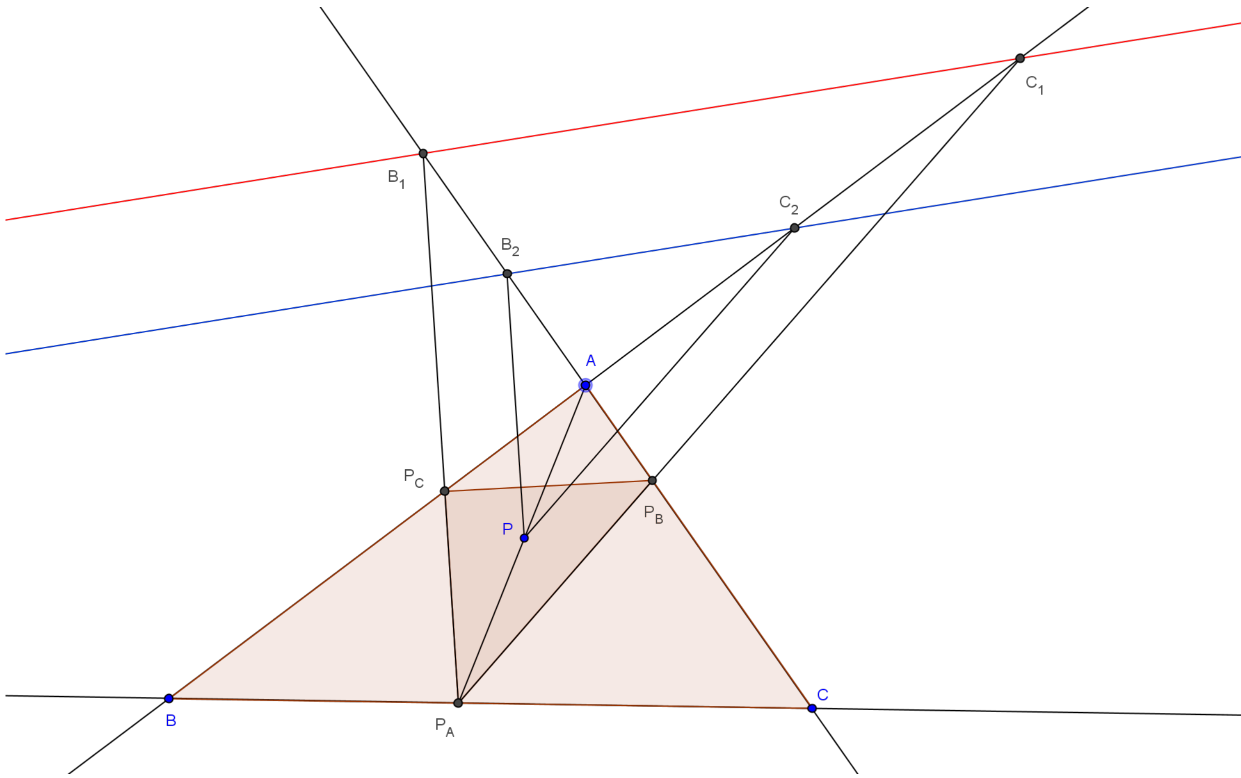

Figure 1

Next, we will prove (2). Let A_B be the intersection of line BC and the the line through P' that is parallel to P_AP_B. View Figure 3 for the labeling of simarly defined points such as B_A.

Lemma 1: WLOG M_A is the midpoint of line segment A_BA_C.

Proof: Let L be the line through A that is parallel to line BC. Let X be the intersection of lines P_AP_C and L, let Y be the intersection of the line through P parallel to P_AP_C and line L, let Z be the intersection of the line through P parallel to P_AP_B and line L, and let W be the intersection lines P_AP_B and L. We can see that $\triangle P_CXA \sim \triangle P_CP_AB \rightarrow \frac{XA}{AP_C} = \frac{P_CB}{BP_A} \rightarrow XA = \frac{(AP_C)(BP_A)}{P_CB}$. A similar analysis will show that $AW = \frac{(AP_B)(P_AC)}{P_BC}$. Noting that $\triangle P_AP_BP_C$ is a cevian triangle, and hence satisfies ceva's theorem, it can be seen that $XA = AW$. Noting that $\triangle AYP \sim \triangle AXP_A \rightarrow AY = \frac{(AX)(AP)}{AP_A} = \frac{(AW)(AP)}{AP_A} = AZ$. Applying a homothety centered at G with a scale factor of $-\frac{1}{2}$ yields the desired result. ■

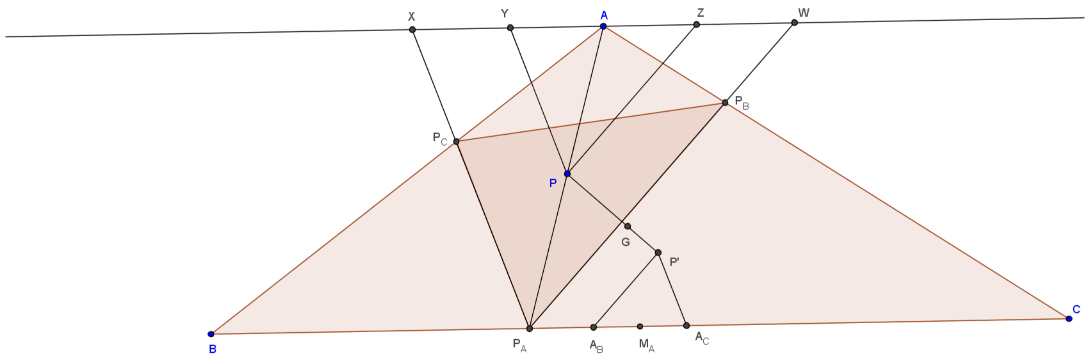

Figure 2

Returning to the main problem at hand, we note that $\triangle CB_3A_C \sim \triangle CB_1P_A \rightarrow CB_3 = \frac{(CB_1)(CA_C)}{CP_A}$. We can now see (4) below.

(4) $\frac{(AB_3)(CA_3)(BC_3)}{(B_3C)(A_3B)(C_3A)} = \frac{(AB_1)(AC_A)(CP_A}{(AP_C)(CB_1)(CA_C)} \cdot \frac{(CA_1)(CB_C)(BP_C)}{(CP_B)(BA_1)(BC_B)} \cdot \frac{(BC_1)(BA_B)(AP_B)}{(BP_A)(AC_1)(AB_A)} = \frac{(AB_1)(CA_1)(BC_1)}{(B_1C)(A_1B)(C_1A)} \cdot \frac{(CP_A)(BP_C)(AP_B)}{(P_AB)(P_CA)(P_BC)} \cdot \frac{(AC_A)(BA_B)(CB_C)}{(BC_B)(CA_C)(AB_A)} = -1$

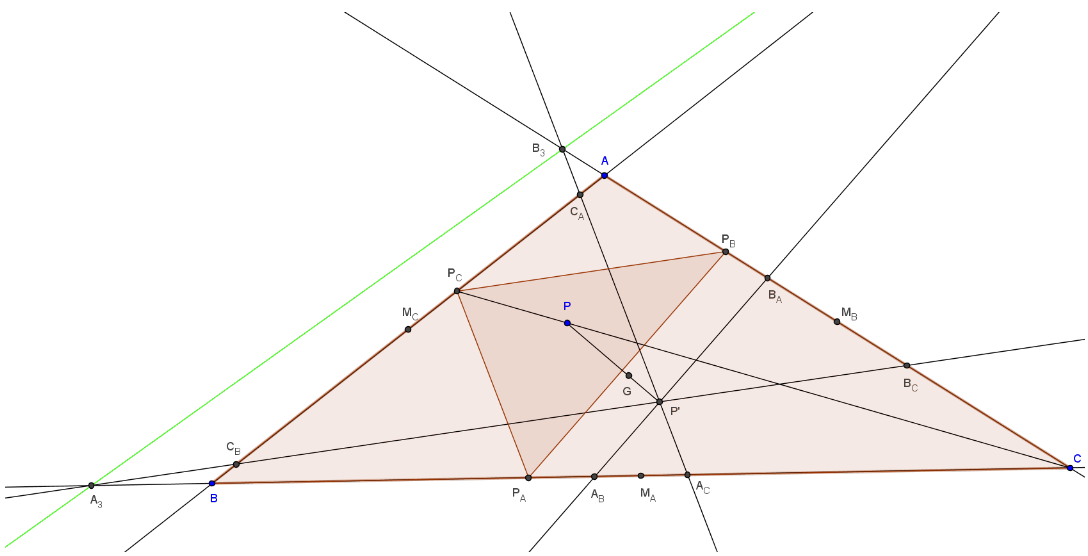

Figure 3

The desired result now follows from menelaus' theorem.■

Lastly, we will prove (3). From the definition of complement we have that $\frac{A_4G}{GP_A} = \frac{M_AG}{GA} = \frac{1}{2} \rightarrow \frac{A_4M_A}{AP_A} = \frac{1}{2}$, so if we allow line AA_4 intersect line BC are A_5, we can see that $P_AM_A = M_AA_5$, which proves the desired result. ■