**Sohail Farhangi** Research statement October, 2021

I have a variety of research interests that are joined by the study and applications of ergodic theory. My results strengthen commonly used tools of ergodic theory, contribute to the understanding of partition regularity of systems of polynomial equations - an important topic in Ramsey theory, and have applications to the theory of uniform distribution, number theory, and the algebra of the Stone-Čech compactification of a semigroup. In the following sections I will describe my current interests, results, ideas for future research, and views on research and collaboration as indicated by the table of contents below. Since my goal in this document is to convey the general flavor of my research I will ocassionally use undefined terms in quotation marks so that I can focus on the broader picture without getting overly technical. Entries in brown are about works in progress.

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# 1 Pointwise ergodic theorems for higher levels of mixing

A measure preserving system (m.p.s.) $(X, \mathscr{B}, \mu, T)$ is a probability space $(X, \mathscr{B}, \mu)$ along with a measure preserving transformation $T : X \to X$. A classical result about measure preserving systems is Birkhoff's Ergodic Theorem.

**Theorem 1.1** (G. Birkhoff)**.** *If $(X, \mathscr{B}, \mu, T)$ is a measure preserving system, $f \in L^1(X, \mu)$, and $\mathcal{I} \subseteq \mathscr{B}$ is the $\sigma$-algebra of $T$-invariant sets, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \mathbb{E}[f|\mathcal{I}] \tag{1}$$

*with convergence taking place pointwise a.e.*

It is known that if $T$ possesses mixing properties then the conclusion of Theorem 1.1 can be strengthened, so let us now recall the ergodic hierarchy of mixing.

**Definition 1.2.** *Let $(X, \mathscr{B}, \mu, T)$ be a m.p.s.*

1. *$T$ is* **ergodic** *if for every $A, B \in \mathscr{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}B) = \mu(A)\mu(B). \tag{2}$$

2. *$T$ is* **weakly mixing** *if for every $A, B \in \mathscr{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| = 0. \tag{3}$$

3. *$T$ is* **mildly mixing** *if for every $A, B \in \mathscr{B}$ we have*

$$IP^* - \lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B). \tag{4}$$

4. *$T$ is* **strongly mixing** *if for every $A, B \in \mathscr{B}$ we have*

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B). \tag{5}$$

5. *$T$ is* **K-mixing** *if for every $A, B \in \mathscr{B}$ we have*

$$\lim_{N\to\infty} \sup_{C\in\sigma(B,N)} |\mu(A \cap T^{-n}C) - \mu(A)\mu(C)| = 0, \tag{6}$$

*where $\sigma(B, N)$ is the $\sigma$-algebra generated by $\{T^{-n}B\}_{n=N}^{\infty}$.*

6. *$T$ is* **Bernoulli** *if there exists $A \in \mathscr{B}$ for which $\mu(A \cap T^{-n}A) = \mu(A)^2$ for every $n \geq 1$.*

It is well known that if $(X,\mathscr{B},\mu,T)$ is an ergodic m.p.s. then $\mathcal{I}=\{A\in\mathscr{B}\mid\mu(A)\in\{0,1\}\}$, which implies the following result.

**Theorem 1.3.** *If $(X,\mathscr{B},\mu,T)$ is an ergodic m.p.s. and $f\in L^1(X,\mu)$, then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^nx)=\int_X f d\mu. \tag{7}$$

*with convergence taking place pointwise a.e.*

N. Wiener and A. Wintner managed to obtain a sharper result when working with a weakly mixing m.p.s.

**Theorem 1.4** (Wiener-Wintner, [45]). *Let $(X,\mathscr{B},\mu,T)$ be a weakly mixing m.p.s. and let $f\in L^1(X,\mu)$. There exists $X'\in\mathscr{B}$ with $\mu(X')=1$, such that for every $x\in X'$ and any $\lambda\in\mathbb{C}$ with $|\lambda|=1$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^nx)\lambda^n=\begin{cases}0 & \text{if } \lambda\neq 1\\ \int_X f d\mu & \text{if } \lambda=1\end{cases}. \tag{8}$$

In [20] the following generalizations of Birkhoff's Pointwise Ergodic Theorem for weakly mixing and strongly mixing measure preserving systems are proven.

**Theorem 1.5** (cf. Theorem 2.6 in [20]). *Let $(X,\mathscr{B},\mu,T)$ be a weakly mixing m.p.s. and let $f\in L^1(X,\mu)$ satisfy $\int_X f d\mu=0$. For a.e. $x\in X$, $(f(T^nx))_{n=1}^{\infty}$ is a "weakly mixing sequence".*

**Theorem 1.6** (cf. Theorem 3.2 in [20]). *Let $(X,\mathscr{B},\mu,T)$ be a strongly mixing m.p.s. and let $f\in L^1(X,\mu)$ satisfy $\int_X f d\mu=0$. For a.e. $x\in X$, $(f(T^nx))_{n=1}^{\infty}$ is a "strongly mixing sequence".*

While weakly mixing sequences and strongly mixing sequences are defined in Section 1 of [20] (motivated by Definition 3.14 in [35]) the definitions are rather technical so we will instead state some of the implications of the previous two theorems.[1] Firstly, let us show that Theorem 1.4 is inplied by Theorem 1.5. A corollary of Theorem 1.5 is

**Corollary 1.7.** *Let $(X,\mathscr{B},\mu,T)$ be a weakly mixing m.p.s. and let $f\in L^1(X,\mu)$ satisfy $\int_X f d\mu=0$. There exists $X'\subseteq X$ with $\mu(X')=1$ such that for every $x\in X'$ and any "compact sequence" $(c_n)_{n=1}^{\infty}$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(T^nx)c_n=0. \tag{9}$$

[1] We would also like to warn the reader that the definition of weakly mixing sequence introduced in [35] is different from the definition of weakly mixing sequences introduced in [3]. Furthermore, in Section 2 we will be mentioning the latter definition.

A bounded sequence $(c_n)_{n=1}^\infty$ of complex numbers is *compact* (Definition 1.4 in [20] which is motivated by Definition 3.16 in [35]) if $\{(c_{n+h})_{n=1}^\infty \mid h \in \mathbb{N}\}$ is precompact in the topology induced by a seminorm that is similar to the Besicovitch seminorm. We note that $(\lambda^n)_{n=1}^\infty$ is a compact sequence for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Furthermore, if $A \subseteq \mathbb{N}$ is of the form $A = \cup_{m=1}^\infty [a_m, b_m]$ with $b_m - a_m \to \infty$, then the sequence $(\mathbb{1}_A(n))_{n=1}^\infty$ is compact. Since the space of compact sequences is closed under pointwise addition and pointwise multiplication, many other examples of compact sequences can be constructed to demonstrate that Corollary 1.7 (and hence Theorem 1.5) is a generalization of the Wiener-Wintner Theorem. Similarly, we can better understand Theorem 1.6 through the following application.

**Corollary 1.8.** *Let $(X, \mathscr{B}, \mu, T)$ be a strongly mixing*[2] *m.p.s. and let $f \in L^1(X, \mu)$ satisfy $\int_X f d\mu = 0$. There exists $X' \subseteq X$ with $\mu(X') = 1$ such that for every $x \in X'$ and any "rigid sequence" $(r_n)_{n=1}^\infty$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) r_n = 0. \tag{10}$$

While we omit the precise definition of rigid sequences, we observe that it includes the class of compact sequences. Furthermore, if $(X, \mathscr{B}, \mu, T)$ is a m.p.s. then $f \in L^2(\mu)$ is *rigid* if there exists a sequence $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ for which $||U_T^{n_k} f - f||_2 \underset{k\to\infty}{\longrightarrow} 0$, and $(X, \mathscr{B}, \mu, T)$ is *rigid* if every $f \in L^2(\mu)$ is rigid. One way we can produce rigid sequences is to take a generic $x \in X$ and some rigid $f \in L^2(\mu)$ since the sequence $(f(T^n x))_{n=1}^\infty$ will be rigid.

My method of proof of Theorems 1.5 and 1.6 can be used to obtain a new pointwise ergodic theorem for ergodic systems, mildly mixing systems, and other systems whose mixing properties can be characterized through filter convergence. Keeping readability and the general audience in mind, these additional results were left as a remark at the end of [20], but they will be included in my thesis.

There are many open ends to the work in [20] that I would like to resolve. One of the questions that I would like to resolve is whether or not such results analogous to Theorems 1.4 and 1.6 can be obtained for $K$-mixing or Bernoulli systems.[3] Another idea that I would like to investigate is whether or not my results can be obtained directly through the use of some maximal inequality so that I don't have to use Birkhoff's Ergodic Theorem in my proofs, and whether or not such methods would yield new results. Additionally, I am interested in proving subsequential pointwise ergodic theorems. A question in this area that I would like to resolve is motivated by the work of J. Bourgain in [14].

**Question 1.9** (cf. Question 2.8 in [20])**.** *If $(X, \mathscr{B}, \mu, T)$ is a weakly mixing m.p.s., $p(x)$ a polynomial with integer coefficients, and $f \in L^r(X, \mu)$ with $r > 1$ is such that $\int_X f d\mu = 0$, then is $(f(T^n x))_{n=1}^\infty$ a weakly mixing sequence for a.e. $x \in X$?*

Lastly, I am interested in determining whether or not there are any analogues of Theorems 1.5 and 1.6 in the context of infinite ergodic theory.

[2] It actually suffices to have $(X, \mathscr{B}, \mu, T)$ be mildly mixing.

[3] I am also curious about whether or not such results exist for more obscure levels of the ergodic hierarchy of mixing such as partial mixing and light mixing.

# 2 Enhancements of van der Corput's difference theorem and connections to the hierarchy of mixing properties of unitary operators

Given a Hilbert space $\mathcal{H}$, a bounded sequence of vectors $(x_n)_{n=1}^{\infty}$ is an **ergodic sequence** (cf. Definition 3.1 in [3]) if

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} x_n|| = 0. \tag{11}$$

Now let us list three variants of the van der Corput's Difference Theorem (abbreviated as vdCDT henceforth) that are commonly used in ergodic theory (cf. [1],[4],[5],[8],[10],[11].[12],[25],[26]).

**Theorem 2.1.** *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors satisfying*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle = 0, \tag{12}$$

*for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is an ergodic sequence.*

**Theorem 2.2.** *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors satisfying*

$$\lim_{h\to\infty} \overline{\lim_{N\to\infty}} \left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0, \tag{13}$$

*then $(x_n)_{n=1}^{\infty}$ is an ergodic sequence.*

**Theorem 2.3.** *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors satisfying*

$$\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H}\overline{\lim_{N\to\infty}} \left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0, \tag{14}$$

*then $(x_n)_{n=1}^{\infty}$ is an ergodic sequence.*

In [19] we examine variants of vdCDT in Hilbert spaces, motivated by the following questions: Why are there so many variants of vdCDT with the same conclusion despite the differing strengths of their assumptions? Why would we ever use Theorems 2.1 and 2.2 when they are both a corollary of Theorem 2.3? We answer these questions by proving a correspondence between many of the known variants of vdCDT and different levels of the ergodic hierarchy of mixing (Definition 2.4). This correspondence could already be seen in [41] for other forms of vdCDT. To better understand this, let us give an alternative description of the ergodic hierarchy of mixing (cf. Definition 1.2).

**Definition 2.4.** *Let $(X, \mathscr{B}, \mu, T)$ be a m.p.s.*

1. *T is* **ergodic** *if for every $A \in \mathscr{B}$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A)=\mu(A)^2. \tag{15}$$

2. *T is* **weakly mixing** *if for every $A \in \mathscr{B}$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\mu(A\cap T^{-n}A)-\mu(A)^2|=0. \tag{16}$$

3. *T is* **mildly mixing** *if for every $A \in \mathscr{B}$ we have*

$$IP^*-\lim_{n\to\infty}\mu(A\cap T^{-n}A)=\mu(A)^2. \tag{17}$$

4. *T is* **strongly mixing** *if for every $A \in \mathscr{B}$ we have*

$$\lim_{n\to\infty}\mu(A\cap T^{-n}A)=\mu(A)^2. \tag{18}$$

5. *T has* **Lebesgue spectrum** *there exists $\{f_j\}_{j=1}^{\infty}\subseteq L^2(\mu)$ for which $\{1\}\cup\{U^nf_j\}_{n\in\mathbb{Z},j\geq 1}$ is an orthogonal basis for $L^2(\mu)$,*

6. *T is* **K-mixing** *if for every $A \in \mathscr{B}$ we have*

$$\lim_{N\to\infty}\sup_{C\in\sigma(A,N)}|\mu(A\cap T^{-n}C)-\mu(A)\mu(C)|=0, \tag{19}$$

*where $\sigma(A,N)$ is the $\sigma$-algebra generated by $\{T^{-n}A\}_{n=N}^{\infty}$.*

7. *T is* **Bernoulli** *if there exists $A \in \mathscr{B}$ for which $\mu(A\cap T^{-n}A)=\mu(A)^2$ for every $n\geq 1$.*

The next three theorems show that Theorems 2.1, 2.2, and 2.3 correspond to Lebesgue spectrum[4], strong mixing, and weak mixing respectively.

**Theorem 2.5** (cf. Corollary 9 in [19])**.** *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors satisfying*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h},x_n\rangle=0, \tag{20}$$

*for every $h\in\mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is a "nearly orthogonal sequence".*

[4] I show that Theorem 2.1 corresponds to elements whose orbit under a unitary operator $U$ generate an orthogonal bases in a Hilbert space, so it is the only variant that is not directly associated with a level of the ergodic hierarchy of mixing, but a loose intuitive connection may be made with Bernoulli systems as follows: If $(X,\mathscr{B},\mu,T)$ is a measure preserving system and for some $A\in\mathscr{B}$ with $\mu(A)>0$ the sequence $(\mathbb{1}_{T^nA}(x))_{n=1}^{\infty}$ satisfies the hypothesis of Theorem 2.1, then $(X,\mathscr{B},\mu,T)$ is a Bernoulli system.

**Theorem 2.6** (cf. Corollary 7 in [19]). *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors satisfying*

$$\lim_{h\to\infty}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h},x_n\rangle\right|=0, \tag{21}$$

*then $(x_n)_{n=1}^{\infty}$ is a "nearly strongly mixing sequence".*

**Theorem 2.7** (cf. Corollary 3 in [19]). *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors satisfying*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h},x_n\rangle\right|=0, \tag{22}$$

*then $(x_n)_{n=1}^{\infty}$ is a "nearly weakly mixing sequence".*

We apologize to the reader for not giving a precise definition of nearly orthogonal, nearly weakly mixing, and nearly strongly mixing sequences (cf. Definition 1.2 in [19]). We point out that nearly weakly mixing and nearly strongly mixing sequences are named such since they are strictly weaker notions that those of weakly mixing and strongly mixing sequences introduced in [3] (cf. Section 4 in [19]).[5] To see why the previous three theorems generalize the first three theorems of this section we observe the following containments of classes of sequence:

$$\begin{aligned}&\text{ergodic sequences} \supsetneq \text{nearly weakly mixing sequences}\\ &\qquad\supsetneq \text{nearly strongly mixing sequences} \supsetneq \text{nearly orthogonal sequences}\end{aligned} \tag{23}$$

Furthermore, just as in Section 1 we can better understand Theorems 2.5, 2.6, and 2.7 through the following applications.

**Corollary 2.8.** *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors satisfying*

$$\lim_{h\to\infty}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h},x_n\rangle\right|=0, \tag{24}$$

*then for any "rigid sequence" of bounded vectors $(r_n)_{n=1}^{\infty}$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_n,r_n\rangle=0. \tag{25}$$

**Corollary 2.9** (cf. Corollary 3 in [19]). *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors satisfying*

[5] In light of the warning in footnote 1, we point out to the reader that the definition of weakly mixing sequence in [35] is almost the same as the definition of nearly weakly mixing sequence in [19].

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h},x_n\rangle\right|=0, \tag{26}$$

*then for any "compact sequence" of bounded vectors* $(c_n)_{n=1}^{\infty}$ *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_n,c_n\rangle=0. \tag{27}$$

We remark that in the case of $\mathcal{H}=\mathbb{C}$ nearly weakly (strongly) mixing sequences are the same as weakly (strongly) mixing sequences introduced in [20], so we refer the reader to Section 1 for examples. We now state an application of each of Corollaries 2.8 and 2.9 to measure preserving systems. Our first application, Theorem 2.10, is obtain by using Theorem 2.7 and Corollary 2.9 in Furstenbergs proof of Szemerédi's Theorem in place of Theorem 2.3.

**Theorem 2.10.** *Let* $(X,\mathscr{B},\mu,T)$ *be an ergodic m.p.s., let* $\ell\in\mathbb{N}$*, let* $A\in\mathscr{B}$ *satisfy* $\mu(A)>0$*, and let* $(B_n)_{n=1}^{\infty}\subseteq\mathscr{B}$ *be a "compact sequence" satisfying* $\mu(B_n)=\alpha>0$ *(Such as* $B_n=S^{-n}B$ *for some measure preserving* $S:X\to X$ *that has discrete spectrum). Then,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(B_n\cap T^{-n}A\cap T^{-2n}A\cap\cdots\cap T^{-\ell n}A)>0. \tag{28}$$

This application also illustrates why Furstenberg's proof of Szemerédi's Theorem uses (an alternate form of) Theorem 2.3 instead of Theorem 2.1. It is not a coincidence that the hypothesis of Theorem 2.1 was too strong for Furstenberg to use, it was because he was working with weakly mixing extensions of measure preserving systems that he needed the weakly mixing version of vdCDT. Our next application arises from the use of Theorem 2.5 in place of Theorem 2.1 in the proof of Theorem 2.1 in [5]. We remind the reader that a brief summary of rigidity occurred after Corollary 1.8.

**Theorem 2.11.** *Let* $(X,\mathscr{B},\mu)$ *be a probability space,* $T,S:X\to X$ *measure preserving transformations, and* $p(x)\in\mathbb{Z}[x]$ *have degree at least 2 and satisfy* $p(0)=0$*. Suppose that the system* $(X,\mathscr{B},\mu,S)$ *is totally ergodic and the system* $(X,\mathscr{B},\mu,T)$ *is rigid.*

*(i) For any* $f,g\in L^{\infty}(\mu)$ *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}U_T^nf\cdot U_S^{p(n)}g=\int_X fd\mu\int_X gd\mu. \tag{29}$$

*(ii) For any* $A,B\in\mathscr{B}$ *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A\cap S^{-p(n)}B)=\mu(A)^2\mu(B). \tag{30}$$

Recently a result of a similar nature was obtained in [24]. I would like to see if we can use the enhanced vdCDTs of [19] to answer questions such as the following.

**Question 2.12.** *Let $(X, \mathscr{B}, \mu)$ be a probability space, $T, S : X \to X$ measure preserving transformations, and $p_1(x), p_2(x) \in \mathbb{Z}[x]$ have distinct degrees larger than 1 and satisfy $p_1(0) = p_2(0) = 0$. Suppose that the system $(X, \mathscr{B}, \mu, S)$ is totally ergodic and the system $(X, \mathscr{B}, \mu, T)$ is rigid.*

*(i) Is it true that for any $f, g \in L^\infty(\mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} U_T^{p_1(x)} f \cdot U_S^{p_2(n)} g = \int_X f d\mu \int_X g d\mu? \tag{31}$$

*(ii) Is it true that for any $A, B \in \mathscr{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-p_1(n)} A \cap S^{-p_2(n)} B) = \mu(A)^2 \mu(B)? \tag{32}$$

Using the connection between vdCDT and the ergodic hierarchy of mixing we are able to obtain two new variants of vdCDT corresponding to ergodicity and mild mixing. For simplicity, we only state the former.

**Theorem 2.13** (cf. Corollary 1 in [19])**.** *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^\infty$ is a bounded sequence of vectors satisfying*

$$\lim_{H\to\infty} \overline{\lim_{N\to\infty}} \left| \frac{1}{HN} \sum_{(h,n)\in[1,H]\times[1,N]} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{33}$$

*then $(x_n)_{n=1}^\infty$ is a "completely ergodic sequence".*

The significance of Theorem 2.13 is that the assumptions are even weaker than those of Theorem 2.3 and the sequences produced are still from a proper subclass of ergodic sequences.

The work in [19] is also a contribution to the theory of uniform distribution. We use the discrepancy of a sequence (a measure of how far away a sequence $(x_n)_{n=1}^\infty \subseteq [0, 1]$ is from being uniformly distributed) to create an inner product-like operation on the set $\mathcal{H}$ of sequences $(x_n)_{n=1}^\infty \subseteq [0, 1]$, which allows us to intuitively view $\mathcal{H}$ as a Hilbert space. We then prove as a consequence of the previous results 4 new vdCDTs in the context of uniform distribution, and generalize the original vdCDT by proving a vastly stronger conclusion under the same assumptions. Each of the 5 variants of vdCDT is also shown to produce a sequence $(x_n)_{n=1}^\infty$ in $[0, 1]$ that has mixing properties corresponding to the different levels of the ergodic hierarchy of mixing, and any of these mixing properties implies uniform distribution. Since many sequences are are shown to be uniformly distributed through the use of the classical vdCDT, this work shows that all of these sequences exhibit far stronger properties than just uniform distribution and that there are many other intermediate levels of mixing and uniform distribution to be studied.

We will now precisely state what the discrepancy of a sequence is so that we can provide concrete formulations of some of the aforementioned results.

**Definition 2.14.** *Given a sequence* $(x_n)_{n=1}^N \subseteq [0,1]$, *the discrepancy of* $(x_n)_{n=1}^N$ *is denoted by* $D_N((x_n)_{n=1}^N)$ *and given by*

$$D_N((x_n)_{n=1}^N) := \sup_{0\le a<b\le 1}\left|\frac{1}{N}|\{1\le n\le N \mid x_n\in(a,b)\}|-(b-a)\right|. \tag{34}$$

*For any infinite sequence* $(x_n)_{n=1}^\infty \subseteq [0,1]$ *we define*

$$\overline{D}((x_n)_{n=1}^\infty) := \overline{\lim_{N\to\infty}} D_N((x_n)_{n=1}^N). \tag{35}$$

**Theorem 2.15** (cf. Theorem 3.4 and Corollary 13 in [19])**.** *If* $(x_n)_{n=1}^\infty \subseteq [0,1]$ *satisfies*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^H \overline{D}((x_{n+h}-x_n)_{n=1}^\infty)=0, \tag{36}$$

*then* $(x_n)_{n=1}^\infty$ *is a "wm-sequence". In particular, if* $(n_k)_{k=1}^\infty$ *is a "compact sequence" then* $(x_{n_k})_{k=1}^\infty$ *is uniformly distributed.*

**Theorem 2.16** (cf. Theorem 3.5 and Corollary 15 in [19])**.** *If* $(x_n)_{n=1}^\infty \subseteq [0,1]$ *satisfies*

$$\lim_{h\to\infty}\overline{D}((x_{n+h}-x_n)_{n=1}^\infty)=0, \tag{37}$$

*then* $(x_n)_{n=1}^\infty$ *is a "sm-sequence". In particular, if* $(n_k)_{k=1}^\infty$ *is a "rigid sequence" then* $(x_{n_k})_{k=1}^\infty$ *is uniformly distributed.*

While we have not defined wm-sequences and sm-sequences (Definition 3.3 in [19]), they are related to the notions of nearly weakly mixing sequences and nearly strongly mixing sequences. Theorem 2.15 is already a generalization of Theorem 4.4 in [12]. Observing that a sequence $(x_n)_{n=1}^\infty \subseteq [0,1]$ is uniformly distributed if and only if $\overline{D}((x_n)_{n=1}^\infty)=0$ (Theorem 1.1 in [32]), we obtain the following corollary to Theorem 2.16 which strengthens the classical vdCDT.

**Corollary 2.17.** *If* $(x_n)_{n=1}^\infty \subseteq [0,1]$ *is a sequence for which* $(x_{n+h}-x_n)_{n=1}^\infty$ *is uniformly distributed for every* $h\in\mathbb{N}$, *then* $(x_n)_{n=1}^\infty$ *is a "sm-sequence".*

Corollary 2.17 is somewhat unsatisfying since we began with a stronger assumption than Theorem 2.16 but did not obtain a stronger conclusion. In light of Theorems 2.5 and 2.6 we would expect Corollary 2.17 to produce some sequences that are associated with nearly orthogonal sequences, but Theorem 3.12 in [19] and the discussion that follows on page 36 show that this is not the case. One of the problems that I would like to investigate is how the conclusion of Corollary 2.17 can be improved.

We will now state the analogue of Theorem 2.13 in the context of uniform distribution. The statement is particularly aesthetic since it does not require the use of $\overline{D}$.

**Theorem 2.18.** *If* $(x_n)_{n=1}^\infty \subseteq [0,1]$ *is such that* $(x_{n+h}-x_n)_{n,h\in\mathbb{N}}$ *is uniformly distributed (as a doubly indexed sequence) then* $(x_n)_{n=1}^\infty$ *is an "e-sequence".*

Theorem 2.18 is the ergodic variant of vdCDT in uniform distribution theory, and e-sequences are associated with completely ergodic sequences.

A problem that I would like to resolve is whether or not there exists a vdCDT which corresponds to K-mixing, whether it be in the context of Hilbert spaces or uniform distribution, which leads to Conjecture 2.19. For the sake of simplicity, we will state Conjecture 2.19 in a form that does not require Cesàro averages.

**Conjecture 2.19.** *If $\mathcal{H}$ is a Hilbert space, $x \in \mathcal{H}$ a vector, and $U : \mathcal{H} \to \mathcal{H}$ a unitary operator for which*

$$\lim_{N\to\infty} \sup_{y \in B(x,N)} |\langle x, y\rangle| = 0, \tag{38}$$

*where $B(x,N) = \overline{Span_{\mathbb{C}}(\{U^n x\}_{n=N}^{\infty})} \cap \{y \in \mathcal{H} \mid ||y|| \leq 1\}$, then for any $z \in \mathcal{H}$ we have*

$$\lim_{N\to\infty} \sup_{y \in B(z,N)} |\langle x, y\rangle| = 0, \tag{39}$$

I would also like to obtain new results in recurrence similar to Theorems 2.10 and 2.11 through the use of my enhanced vdCDTs. Lastly, we observe that Corollary 2.17 implies that sm-sequences are prevalent in the theory of uniform distribution, so I would like to better understand what implications this has and what applications can be found from these mixing generalizations of uniform distribution.

# 3 Van der Corput sets and sets of recurrence

We will assume that the reader of this section has first read Section 2. Another way in which one may try to enhance vdCDT is through the study of *van der Corput sets*.

**Definition 3.1** (cf. [30],[39]). *A set $D \subseteq \mathbb{N}$ is a* **van der Corput set (vdC set)** *if it satisfies any one of the following equivalent conditions.*

1. *If $(x_n)_{n=1}^\infty \subseteq [0,1]$ is a sequence for which $(x_{n+h}-x_n)_{n=1}^\infty$ is uniformly distributed for every $h \in D$, then $(x_n)_{n=1}^\infty$ is also uniformly distributed.*

2. *If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^\infty$ is a bounded sequence of vectors in $\mathcal{H}$ satisfying*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h},x_n\rangle = 0, \tag{40}$$

   *for every $h \in D$, then $(x_n)_{n=1}^\infty$ is an ergodic sequence.*

3. *For any Hilbert space $\mathcal{H}$, any unitary operator $U : \mathcal{H} \to \mathcal{H}$, and any $f \in \mathcal{H}$ satisfying $\langle U^d f, f\rangle = 0$ for every $d \in D$, $(U^n f)_{n=1}^\infty$ is an ergodic sequence.*

4. *If $\mu$ is a probability measure on $\mathbb{T}$ for which $\hat{\mu}(d) = 0$ for all $d \in D$, then $\mu(\{0\}) = 0$.*

5. *If $\mu$ is a probability measure on $\mathbb{T}$ for which $\hat{\mu}(d) = 0$ for all $d \in D$, then $\mu$ is continuous.*

6. *If $\mu$ is a probability measure on $\mathbb{T}$ for which $\sum_{d\in D}|\hat{\mu}(d)| < \infty$, then $\mu$ is continuous.*

7. *For each $\epsilon > 0$ there exists a trigonometric polynomial $P_\epsilon : \mathbb{T} \to [-\epsilon,\infty)$ such that $\hat{P}_\epsilon(n) = 0$ if $n \in \mathbb{Z}\setminus(D\cup -D)$ and $P(0) = 1$.*

While Definition 3.1 already shows that vdC sets are related to uniform distribution, operator theory, and harmonic analysis, we would also like to establish a connection with *sets of recurrence*. In order to have a fuller discussion regarding future work, we begin with a seemingly unrelated definition.

**Definition 3.2.** *A* **topological dynamical system** *is a pair $(X,T)$ in which $X$ is a compact Hausdorff space and $T : X \to X$ is a continuous map. The system $(X,T)$ is* **minimal** *if for every $x \in X$ the sequence $\{T^n x\}_{n=1}^\infty$ is dense in $X$.*

**Definition 3.3.** *A set $R \subseteq \mathbb{N}$ is a* **set of measurable recurrence** *if for any m.p.s. $(X,\mathscr{B},\mu,T)$ and any $A \in \mathscr{B}$ with $\mu(A) > 0$ there exists $r \in R$ for which $\mu(A\cap T^{-r}A) > 0$. Similarly, $R \subseteq \mathbb{N}$ is a* **set of topological recurrence** *if for any minimal topological dynamical system $(X,T)$ and any open set $U \subseteq X$, there exists $r \in R$ for which $U\cap T^{-r}U \neq \emptyset$.*

Given a m.p.s. $(X,\mathscr{B},\mu,T)$, one can apply item 3 of Definition 3.1 to the Hilbert space $L^2(X,\mu)$ with the unitary operator $U_T$ induced by $T$ to deduce the following result.

**Theorem 3.4.** *If $D \subseteq \mathbb{N}$ is a vdC set, then $D$ is also a set of measurable recurrence.*

The converse of Theorem 3.4 is due to J. Bourgain [13].

**Theorem 3.5.** *There exists a set of measurable recurrence $R \subseteq \mathbb{N}$ which is not a vdC set.*

One of the reasons that Theorem 3.5 is interesting is because of how difficult it is to prove. Another reason is that item 3 of Definition 3.1 allows us to view vdC sets as sets of recurrence for Hilbert spaces (as in [36]) and Theorem 3.5 says that this is a strictly stronger notion than sets of measurable recurrence. In order to discuss some analogous questions that are still open we first require some more terminology.

**Definition 3.6** (Proposition 2.5 and Definition 10 in [9])**.**

1. *A set $D \subseteq \mathbb{N}$ is an* **enhanced vdC set** *if for any sequence $(x_n)_{n=1}^{\infty}$ of complex numbers of modulus 1 satisfying*

$$\lim_{d\to\infty, d\in D} \overline{\lim_{N\to\infty}} \left| \frac{1}{N} \sum_{n=1}^{N} x_{n+d}\overline{x_n} \right| = 0, \tag{41}$$

   *for every $d \in D$, we have that $(x_n)_{n=1}^{\infty}$ is an ergodic sequence, i.e.,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = 0. \tag{42}$$

2. *A set $D \subseteq \mathbb{N}$ is a* **nice vdC set** *if for any sequence $(x_n)_{n=1}^{\infty}$ of complex numbers of modulus 1 we have*

$$\lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|^2 \leq \lim_{d\to\infty, d\in D} \overline{\lim_{N\to\infty}} \left| \frac{1}{N} \sum_{n=1}^{N} x_{n+d}\overline{x_n} \right| = 0. \tag{43}$$

One interesting open topic that I would like to investigate is the number of equivalent characterizations of enhanced vdC sets and nice vdC sets that can be obtained in analogy with Definition 3.1. Another interesting line of questioning is to what degree we can obtain analogues of Theorems 3.4 and 3.5 for other kinds of vdC sets? To this end we require a few more definitions.

**Definition 3.7.**

1. *A set $R \subseteq \mathbb{N}$ is a* **set of strong recurrence** *if for any m.p.s. $(X, \mathscr{B}, \mu, T)$ and any $A \in \mathscr{B}$ with $\mu(A) > 0$ we have*

$$\overline{\lim_{r\to\infty, r\in R}} \mu(A \cap T^{-r}A) > 0. \tag{44}$$

2. *A set $R \subseteq \mathbb{N}$ is a* **set of nice recurrence** *if for any m.p.s $(X, \mathscr{B}, \mu, T)$, any $A \in \mathscr{B}$ with $\mu(A) > 0$, and any $\epsilon > 0$ there exists $r \in R$ for which $\mu(A \cap T^{-r}A) > \mu(A)^2 - \epsilon$.*

In Proposition 3.5 of [9] it is shown that every enhanced vdC set is a set of strong recurrence and in Question 8 it is asked whether or not every nice vdC set is a set of nice recurrence, which is a question that I have answered in an unpublished manuscript.

**Theorem 3.8** (S. Farhangi, 2018)**.** *Every nice vdC set is also a set of nice recurrence.*

I would like to continue gathering results in this direction by determining whether or not every set of strong recurrence is an enhanced vdC set and by determining whether or not every set of nice recurrence is a nice vdC set.

Another interesting line of inquiry is motivated by [34], which summarizes some of the work in [33] and [23] to show the following inclusions of classes of sets of recurrence.

$$\text{topological recurrence} \supsetneq \text{measurable recurrence} \supsetneq \text{strong recurrence} \supsetneq \text{nice recurrence}. \quad (45)$$

It is still unknown whether or not every vdC set is also an enhanced vdC set and whether or not every enhanced vdC set is also a nice vdC set, and these are some other questions that I would like to resolve. Another interesting research question is whether or not there is a topological analogue of vdC sets and how they fit into the picture?

A completely different line of research that I have initiated and would like to continue is the relationship between vdC sets and the mixing sequences discussed in Section 2. The types of results that I have proven in this direction are illustrated by the following example.

**Theorem 3.9** (S. Farhangi, 2019, cf. Theorem 2.7)**.** *If $D \subseteq \mathbb{N}$ is a vdC set, $\mathcal{H}$ is a Hilbert space, and $(x_n)_{n=1}^{\infty}$ is a bounded sequence of vectors in $\mathcal{H}$ satisfying*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle = 0, \quad (46)$$

*for every $h \in D$, then $(x_n)_{n=1}^{\infty}$ is a "nearly weakly mixing sequence".*

For future research I not only want to better understand the connections between various types of vdC sets and the mixing sequences that they produce, but I would also like find connections between sets of measurable recurrence (and their variants) and mixing sequences since sets of recurrence and vdC sets are so intimately related.

## 4 On the partition regularity of $ax + by = cw^m z^n$

An important goal of modern Ramsey Theory is to better understand which systems of polynomial equations are partition regular over $\mathbb{N}$. While R. Rado [38] determined which finite systems of linear equations are partition regular over $\mathbb{N}$, it is not even known whether or not the equation $x^2 + y^2 = z^2$ is partition regular over $\mathbb{N}$, so it is helpful to find examples of polynomial equations whose partition regularity is known. In [2] and [18] many such examples are provided. The work in [22] provides more examples by expanding upon the results of V. Bergelson ([6], Section 6) and N. Hindman [28], who independently showed that the equation $x + y = wz$ is partition regular over $\mathbb{N}$. We generalize this result by attempting to characterize the values of $a, b, c \in \mathbb{Z} \setminus \{0\}$ and $m, n \in \mathbb{N}$ for which the equation $ax + by = cw^m z^n$ is partition regular over $\mathbb{Z} \setminus \{0\}$. Using a result of V. Bergelson ([5], page 53) as well as the nonlinear Rado conditions from [2] we establish the following result.

**Theorem 4.1** (Theorem 1.1(i) in [22]). *If $a, b, c \in \mathbb{Z} \setminus \{0\}$ and $m, n \in \mathbb{N}$ are such that $m, n \geq 2$, then the equation*

$$ax + by = cw^m z^n \tag{47}$$

*is partition regular over $\mathbb{Z} \setminus \{0\}$ if and only if $a + b = 0$.*

Theorem 4.1 reduces our study to the case of $m = 1$, so we prove numerous other results that all support Conjecture 4.2.

**Conjecture 4.2** (Conjecture 8.1 in [22]). *Given $a, b, c \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$, the equation*

$$ax + by = cwz^n \tag{48}$$

*is partition regular over $\mathbb{Z} \setminus \{0\}$ if and only if one of $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a+b}{c}$ is an nth power in $\mathbb{Q}$.*

In particular, we show that Conjecture 4.2 holds when $n$ is odd. We then investigate the partition regularity of $ax + by = cwz^n$ over general integral domains $R$ and obtain the following positive result.

**Theorem 4.3** (Theorem 6.3 in [22]). *Let $R$ be an integral domain with field of fractions $K$. If $a, b, c \in R \setminus \{0\}$ and $n \in \mathbb{N}$ are such that one of $\frac{a}{c}$, $\frac{b}{c}$, or $\frac{a+b}{c}$ is an nth power in $K$, then the equation*

$$ax + by = cwz^n \tag{49}$$

*is partition regular over $R \setminus \{0\}$.*

A particularly aesthetic result arises when we set $n = 1$ in Theorem 4.3.

**Corollary 4.4.** *Let $R$ be an integral domain and let $a, b, c \in R \setminus \{0\}$ be arbitrary. The equation*

$$ax + by = cwz \tag{50}$$

*is partition regular over $R \setminus \{0\}$.*

To prove Theorem 4.3 we obtain and utilize new results of independent interest about the algebra of the Stone-Čech compactification of an integral domain $R$ (Theorems 9.18, 9.25, and 9.28 in [22]). We also investigate the partition regularity of systems of equations of the form $a_ix_i + b_iy_i = c_iw_iz_i^n, 1 \le i \le k$ over a general integral domain $R$, and in the case of $R = \mathbb{Z}$ we obtain results that support Conjecture 4.5.

**Conjecture 4.5** (Conjecture 8.6 in [22]). *Let $a_1, \cdots, a_k, b_1, \cdots, b_k, c_1, \cdots, c_k \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$. The system of equations*

$$\begin{array}{ccccc} a_1x_1 & + & b_1y_1 & = & c_1w_1z_1^n \\ & & \vdots & & \\ a_kx_k & + & b_ky_k & = & c_kw_kz_k^n \end{array} \tag{51}$$

*is partition regular over $\mathbb{Z} \setminus \{0\}$ if and only if*

$$\mathbb{Q} \cap \bigcap_{i=1}^{k} \left\{ \sqrt[n]{\frac{a_i}{c_i}}, \sqrt[n]{\frac{b_i}{c_i}}, \sqrt[n]{\frac{a_i + b_i}{c_i}} \right\} \neq \emptyset \tag{52}$$

To discuss the negative results that we obtain we introduce the following notation: If $r, s \in \mathbb{Z}$, and $p \in \mathbb{N}$ is a prime for which $p \nmid s$, then we define $\frac{r}{s} \equiv rs^{-1} \pmod p$. The main method that we use to show that the equation $ax + by = cwz^n$ is not partition regular over $\mathbb{Z} \setminus \{0\}$ for some values of $a, b, , c$, and $n$ is to find a prime $p$ modulo which none of $\frac{a}{c}, \frac{b}{c}$, and $\frac{a+b}{c}$ are $n$th powers and then use the classical $c_p$ partition of Rado. To see the effectiveness of this method, we now state one of the number theoretic results that we obtained, which is a partial generalization of the Grunwald-Wang Theorem ([27],[43],[44]).

**Theorem 4.6** (Corollary 4.9 in [22]). *Let $\alpha, \beta, \gamma \in \mathbb{Q} \setminus \{0\}$.*

*(i) Suppose $n$ is odd and $\alpha, \beta, \gamma$ are not $n$th powers; or*

*(ii) Suppose $n$ is even, $\alpha, \beta, \gamma$ are not $\frac{n}{2}$th powers, and $\alpha$ is not an $\frac{n}{4}$th power if $4 \mid n$.*

*Then there exists infinitely many primes $p \in \mathbb{N}$ modulo which none of $\alpha, \beta, \gamma$ are $n$th powers.*

In Section 8 of [22] we mention many concrete instances of Conjectures 4.2 and 4.5 that remain unresolved and lay a foundation for future work. For example, we observe that the partition regularity of $16x+17y = wz^8$ is not known. Another natural direction for extending these results is to increase the number of variables and/or to continue working in general integral domains as in the following question.

**Question 4.7** (Question 8.4 in [22]). *Given an integral domain $R$, $r_1, \cdots, r_k, c \in R \setminus \{0\}$, and $n_1, \cdots, n_s \in \mathbb{N}$, when is*

$$\sum_{i=1}^{k} r_ix_i = c\prod_{j=1}^{s} y_j^{n_j} \tag{53}$$

*partition regular over $R \setminus \{0\}$?*

Lastly, I am also interested in investigating the partition regularity of $ax + by = cw^m z^n$ over more general rings such $\bigoplus_{n=1}^{\infty}(\mathbb{Z}/p\mathbb{Z})$ and the quaternions. The foundation for future work in this direction lays in Theorem 9.25 of [22] which applies to a wide class of division rings, as well as in [7] and [16] where results are obtained about partition regularity of systems of linear equations over rings with zero divisors.

## 5 Distance Graphs and Arithmetic Progressions

A classical result in Ramsey Theory is the following theorem of van der Waerden [42].

**Theorem 5.1.** *For any $r, \ell \in \mathbb{N}$ and any partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ there exists $a, d \in \mathbb{N}$ and $1 \le i_0 \le r$ such that $\{a + id\}_{i=0}^{\ell} \subseteq C_{i_0}$.*

One way in which we can strengthen Theorem 5.1 is to place restrictions on the values of $d$ as is done in the next result.

**Theorem 5.2** (cf. [8])**.** *For any $r, \ell \in \mathbb{N}$ and any partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ there exists $a, m \in \mathbb{N}$ and $1 \le i_0 \le r$ such that $\{a + im^2\}_{i=0}^{\ell} \subseteq C_{i_0}$.*

An important goal in modern Ramsey theory is to better understand results analogous to Theorem 5.2, so we introduce the following terminology.

**Definition 5.3** (cf. Section 1 in [15])**.** *Given $r \in \mathbb{N}$, a set $D \subseteq \mathbb{N}$ is said to be* **r-large** *if for any partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ and any $\ell \in \mathbb{N}$ there exists $a \in \mathbb{N}, d \in D$, and $1 \le i_0 \le r$ for which $\{a + id\}_{i=0}^{\ell} \subseteq C_{i_0}$. If $D$ is $r$-large for every $r \in \mathbb{N}$ then $D$ is* **large**.

Using the language above, van der Waerden's Theorem says that $D = \{m\}_{m=1}^{\infty}$ is a large set and Theorem 5.2 says that $D = \{m^2\}_{m=1}^{\infty}$ is a large set. Determining which subsets $\mathbb{N}$ are large is a topic of great interest, which leads to the following question from [15].

**Question 5.4.** *If $D \subseteq \mathbb{N}$ is 2-large, must $D$ also be large?*

In [15] it was shown that if $D$ is a lacunary sequence, then it is not a large set, and it was asked if a lacunary sequence could be a 2-large set. In [21] we answer this question in the negative through the use of dynamics. To describe this in more detail we require the following definition.

**Definition 5.5** (cf. Definitions 2 and 3 in [31])**.** *A set $D \subseteq \mathbb{N}$ is a* **Bohr(n) recurrent** *if for any $(\alpha_1, \cdots, \alpha_n) \in \mathbb{T}^n$ any any $\epsilon > 0$ there exists $d \in D$ for which $||d\alpha_i|| < \epsilon$ for $1 \le i \le n$. If $D$ is Bohr(n) recurrent for every $n \in \mathbb{N}$ then $D$ is* **Bohr recurrent**.

It is known that any large set is a Bohr recurrent set, so a step towards resolving Question 5.4 is to determine whether or not any 2-large set is also a Bohr recurrent set, which leads us to our next result.

**Theorem 5.6** (Theorem 2 in [21], Lemma 7 in [29])**.** *If $D \subseteq \mathbb{N}$ is 2-large then $D$ is Bohr(1) recurrent.*

Combining Theorem 5.6 with the fact that lacunary sequences are not Bohr(1) recurrent (see any of [17],[31],[37],[40]) we immediately deduce that lacunary sequences are not 2-large.

A question that I would like to investigate in the future is whether or not 2-large sets are Bohr(n) recurrent for values of $n > 1$. Since it is known that every large set is a set of topological recurrence and every set of topological recurrence is Bohr recurrent, another natural step towards answering Question 5.4 would be to determine whether or not 2-large sets are sets of topological recurrence. I am also interested in the famous open problem of determining whether or not Bohr recurrent sets are also sets of topological recurrence (cf. Section 4 in [31]),

# 6 Dynamics and Canonical Ramsey Theory

The classical theorem of van der Waerden about arithmetic progressions (cf. Theorem 5.1) is known to be equivalent to the following result in dynamics.

**Theorem 6.1** (Dynamical van der Waerden)**.** *For any minimal topological dynamical system $(X, T)$, any nonempty open set $U \subseteq X$, and any $\ell \in \mathbb{N}$ there exists $d \in \mathbb{N}$ for which*

$$U \cap T^{-d}U \cap T^{-2d}U \cap \cdots \cap T^{-\ell d}U \neq \emptyset. \tag{54}$$

Many other results in partition Ramsey theory have dynamical formulations similar to Theorem 6.1 which allows us to use the machinery of topological dynamics to obtain combinatorial results. Similarly, the Furstenberg correspondence prinicipal lets us use the machinery of measure theory and measurable dynamics to obtain results in density Ramsey theory.

**Theorem 6.2** (Furstenberg Correspondence Principle)**.** *Given a set $E \subseteq \mathbb{Z}$ of positive upper Banach density, there exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ such that $\mu(A) = d^*(E)$ and*

$$d^*(E \cap (E - n_1) \cap \cdots \cap (E - n_k)) \geq \mu(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_k}A) \tag{55}$$

*for any $n_1, \cdots, n_k \in \mathbb{N}$.*

As of now, there is no such connection between canonical Ramsey theory and dynamics. The purpose of this section is to state conjectures in dynamics that yield results in canonical Ramsey theory. To this end, we begin by stating the canonical van der Waerden Theorem and what the author refers to as the canonical Szemerédi Conjecture.

**Theorem 6.3** (Canonical van der Waerden)**.** *For any $k \in \mathbb{Z}$ and any coloring $\chi : \mathbb{Z} \to \mathbb{N}$, there exists $a, d \in \mathbb{N}$ such that either*

*(i) $\{a + id\}_{i=0}^{k}$ is monochromatic under $\chi$, or*

*(ii) $\{a + id\}_{i=0}^{k}$ is rainbow under $\chi$.*

**Conjecture 6.4** (Canonical Szemerédi)**.** *For any $A \subseteq \mathbb{Z}$ and $k \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that either*

*(i) $A \cap (A - d) \cap \cdots \cap (A - kd) \neq \emptyset$, or*

*(ii) $(A - id) \cap (A - jd) = \emptyset$ for every $0 \leq i < j \leq k$.*

Since canonical Ramsey theory allows for an infinite number of colors, this intuitively corresponds to dynamics on locally compact Hausdorff spaces. Since we can choose to work in the one-point compactification of such spaces, we do so and lose minimality of our dynamical system instead of compactness.

**Conjecture 6.5** (Dynamical Canonical van der Waerden)**.** *For any compact Hausdorff topological space $X$, any homeomorphism $T : X \to X$, any open set $U \subseteq X$, and any $k \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that either* *(i)* $U \cap T^d U \cap \cdots \cap T^{kd} U \neq \emptyset$, *or*

*(ii)* $T^{id} U \cap T^{jd} U = \emptyset$ *for every* $0 \leq i < j \leq k$.

**Theorem 6.6.** *Conjecture 6.5 for metric spaces implies the Canonical van der Waerden Theorem.*

Surprisingly, Conjecture 6.5 also implies Conjecture 6.4 (and Szemerédi's Theorem) despite the apparent lack of measurable dynamics!

**Theorem 6.7.** *Conjecture 6.5 implies the Canonical Szemerédi Theorem.*

Does this mean that Conjecture 6.5 is significantly easier to prove for metric spaces since that does not necessarily imply Szemerédi's Theorem as a corollary? The strength of Conjecture 6.5 for compact Hausdorff spaces can also be seen by the fact that it implies the following strengthening of the canonical van der Waerden Theorem.

**Theorem 6.8.** *Assume that Conjecture 6.5 is true. Then for any* $k \in \mathbb{N}$ *and any coloring* $\chi : \mathbb{Z} \to \mathbb{N}$, *either*

*(i)* *there exists* $a, d \in \mathbb{N}$ *for which* $\{a + id\}_{i=0}^{k}$ *is monochromatic under* $\chi$, *or*

*(ii)* *there exists* $d \in \mathbb{N}$ *such that for any color* $m \in \mathbb{N}$ *and any* $0 \leq i < j \leq k$ *we have*

$$(\chi^{-1}(m) - id) \cap (\chi^{-1}(m) - jd) = \emptyset. \tag{56}$$

Theorem 6.8 tells us that in any (possibly infinite) coloring of the integers, there either exists a monochromatic arithmetic progression of length $k+1$, or there exists an integer $d$ such that any arithmetic progression of length $k+1$ with common difference $d$ is rainbow. Now let us examine some conjectures in measurable dynamics that resemble Furstenberg's multiple recurrence Theorem and imply the canonical Szemerédi Theorem.

**Theorem 6.9** (Dynamical Canonical Szemerédi)**.** *Statements* $(1)-(4)$ *are equivalent and they each imply Conjecture 6.4.*

*(1)* *For any invertible measure preserving system* $(X, \mathscr{B}, \mu, T)$ *with* $X$ $\sigma$*-finite, any* $A \in \mathscr{B}$, *and any* $k \in \mathbb{N}$, *there exists* $d \in \mathbb{N}$ *such that either*

  *(i)* $\mu(A \cap T^d A \cap \cdots \cap T^{kd} A) > 0$, *or*

  *(ii)* $\mu(T^{id} A \cap T^{jd} A) = 0$ *for every* $0 \leq i < j \leq k$.

*(2)* *Let* $(\mathbb{R}, \mathscr{B}, m, T)$ *be an invertible measure preserving system in which* $\mathscr{B}$ *is the Lebesgue* $\sigma$*-algebra and* $m$ *is a the Lebesgue measure. For any* $A \in \mathscr{B}$ *and* $k \in \mathbb{N}$, *there exists* $d \in \mathbb{N}$ *such that either*

  *(i)* $\mu(A \cap T^d A \cap \cdots \cap T^{kd} A) > 0$, *or*

  *(ii)* $\mu(T^{id} A \cap T^{jd} A) = 0$ *for every* $0 \leq i < j \leq k$.

*(3) For any invertible nonsingular system* $(X, \mathscr{B}, \mu, T)$ *with* $\mu$ *a probability measure, any* $A \in \mathscr{B}$*, and any* $k \in \mathbb{N}$*, there exists* $d \in \mathbb{N}$ *such that either*

- *(i)* $\mu(A \cap T^d A \cap \cdots \cap T^{kd} A) > 0$*, or*
- *(ii)* $\mu(T^{id} A \cap T^{jd} A) = 0$ *for every* $0 \leq i < j \leq k$*.*

*(4) For any invertible nonsingular system* $([0,1], \mathscr{B}, m, T)$ *in which* $\mathscr{B}$ *is the Lebesgue* $\sigma$*-algebra and* $m$ *is the Lebesgue measure, any* $A \in \mathscr{B}$*, and any* $k \in \mathbb{N}$*, there exists* $d \in \mathbb{N}$ *such that either*

- *(i)* $\mu(A \cap T^d A \cap \cdots \cap T^{kd} A) > 0$*, or*
- *(ii)* $\mu(T^{id} A \cap T^{jd} A) = 0$ *for every* $0 \leq i < j \leq k$*.*

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