Inequalities and Quadratic Forms

by Sohail Farhangi

In this paper we will investigate a method of transforming inequalities using quadratic forms.

Problem: Given real positive number a, b, and c, prove the following inequality, and find when equality holds.

(1) $a\sqrt{4a^2+5bc}+b\sqrt{4b^2+5ca}+c\sqrt{4c^2+5ab} \geq (a+b+c)^2$.

Solution: We notice that (1) is equivalent to (2) below.

(2) $a\sqrt{4a^2+5bc}-a^2-ab-ac+b\sqrt{4b^2+5ca}-ab-b^2-bc+c\sqrt{4c^2+5ab}-ca-bc-c^2 \geq 0$.

Now we define the following variables so that we may restate (2) as (3).

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$A = \begin{pmatrix} \sqrt{4+5\frac{bc}{a^2}}-1 & -1 & -1 \\ -1 & \sqrt{4+5\frac{ca}{b^2}}-1 & -1 \\ -1 & -1 & \sqrt{4+5\frac{ab}{c^2}}-1 \end{pmatrix}$$

(3) $v^t A v \geq 0$

We will now show that A is a positive semi-definite matrix, then we will only need to find the eigenvectors corresponding to an eigenvalue of 0 to finish the problem. To show that the hermitian matrix A is positive semi-definite, we only need to verify that all principal minors are nonnegative. It is clear that all $1x1$ minors of A are positive. Next, if we note that $\sqrt{4+5\frac{bc}{a^2}}-1 \geq \sqrt{4+5*0}-1 = 1$, it is easily seen that all $2x2$ minors of A are nonnegative. Lastly, we need to verify that the determinant of A is nonnegative, and this is equivalent to (4) below.

$(\sqrt{4+5\frac{bc}{a^2}}-1)[(\sqrt{4+5\frac{ca}{b^2}}-1)(\sqrt{4+5\frac{ab}{c^2}}-1)-1]-(-1)[-(\sqrt{4+5\frac{ab}{c^2}}-1)-1]+(-1)[1-(-1)(\sqrt{4+5\frac{ca}{b^2}}-1)] \geq 0 \Leftrightarrow$

$\sqrt{4+5\frac{bc}{a^2}}\sqrt{4+5\frac{ca}{b^2}}\sqrt{4+5\frac{ab}{c^2}} \geq \sqrt{4+5\frac{bc}{a^2}}\sqrt{4+5\frac{ca}{b^2}}+\sqrt{4+5\frac{ca}{b^2}}\sqrt{4+5\frac{ab}{c^2}}+\sqrt{4+5\frac{ab}{c^2}}\sqrt{4+5\frac{bc}{a^2}} \geq 0 \Leftrightarrow$

(4) $1 \geq (4+5\frac{bc}{a^2})^{-\frac{1}{2}}+(4+5\frac{ca}{b^2})^{-\frac{1}{2}}+(4+5\frac{ab}{c^2})^{-\frac{1}{2}}$

We note that (4) is a homogeneous inequality, so we may impose the condition $abc = 1$ to obtain (5).

(5) $1 \geq (4+5a^{-3})^{-\frac{1}{2}} + (4+5b^{-3})^{-\frac{1}{2}} + (4+5c^{-3})^{-\frac{1}{2}}$

We may now perform the following substitution. $x = a^{-3}, y = b^{-3}$, and $z = c^{-3}$ to obtain (6).

(6) $1 \geq (4+5x)^{-\frac{1}{2}} + (4+5y)^{-\frac{1}{2}} + (4+5z)^{-\frac{1}{2}}$, $xyz = 1$.

(6) is a well known inequality with the 2 equality cases, so A is positive semi-definite as desired. The proof of (6) is being ommitted as it detracts from the main method being illustrated in this paper. We will now consider the 2 equality cases in order to find when equality holds in (1).

Case 1: $x = y = z = 1$. In this case it is evident that $a = b = c$, and plugging these values in A yields the matrix A_1 below.

$$A_1 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

It can be seen that the eigenvalues of A_1 are $0, 3$, and 3, so the eigenspace corresponding to the eigenvalue 0 has dimension 1. We can see that

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector of A_0 corresponding to the eigenvalue of 0, and it follows that equality in (1) holds if $a = b = c$.

Case 2: $\{x, y, z\} = \lim_{n\to\infty}\{\frac{1}{n}, \frac{1}{n}, n^2\}$. For short hand, we will denote this solution as $\{x, y, z\} = \{0, 0, \infty\}$. Moreover, it can be seen that there are 3 permutations of this solution, but it is only necessary to consider 1 of them. In this case it is evident that $a = b = \infty$, and $c = 0$. We can see that plugging these values in A yields the matrix A_2 below.

$$A_2 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & \infty \end{pmatrix}$$

It can be see that

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Is an eigenvector corresponding to the eigenvalue 0. Furthermore, we can see that v_1 is the only eigenvector corresponding to the eigenvalue 0 as follows. Consider the vector v_2 below.

$$v_2 = \begin{pmatrix} r \\ s \\ t \end{pmatrix}$$

In order for v_2 to be an eigenvector corresponding to the eigenvalue 0 of the matrix A_2, we must have $t = 0$ as seen from the fact that the entry in the third row and third column of A_2 is ∞, while the other entries in the third row are finite. Examining the first row of A_2 tells us that $r = s$, so we can see that v_1 is the only eigenvector of A_2 corresponding to the eigenvalue 0 as desired, and this corresponds to the solution $a = b$, and $c = 0$ for (1). ■

Exercise 1

Consider a function of the form $f(a,b,c) = \sqrt{4 + \sum_{i=0}^{n} d_i a^{r_i} b^{s_i} c^{t_i}}$ where $\{d_i\}_{i=0}^{\infty}$, $\{r_i\}_{i=0}^{\infty}$, $\{s_i\}_{i=0}^{\infty}$, and $\{t_i\}_{i=0}^{\infty}$ are all real sequences with $d_i > 0 \ \forall \ 1 \leq i \leq n$, and $f(1,1,1) = 3$. For positive real numbers a, b, and c, and any real number x, y, and z, prove the following inequality given that $abc \geq 1$.

$$x^2 f(a,b,c) + y^2 f(c,a,b) + z^2 f(b,c,a) \geq (x+y+z)^2.$$

What can be said about the equality case? The equality case does have some dependence on the function f.