Complements through Cevian Triangles

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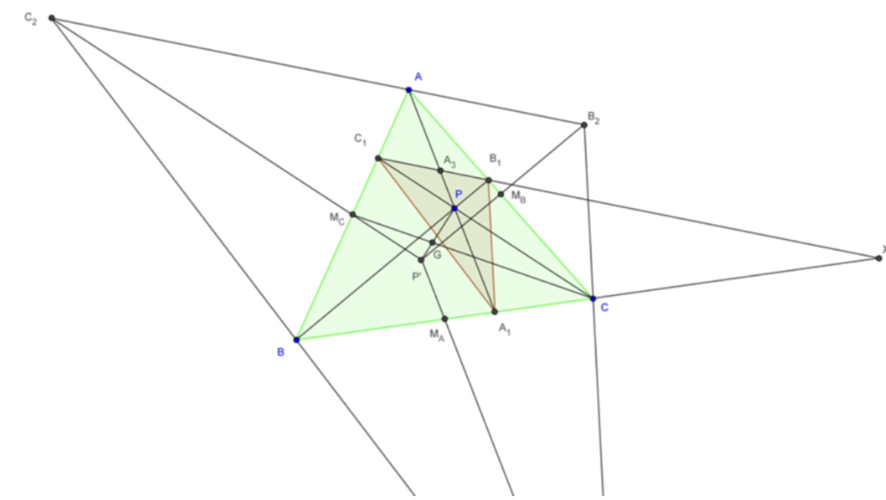
Problem: Given $\triangle ABC$ and a point P inside it, construct point P' as follows. Let lines AP, BP, and CP intersect sides BC, CA, and AB respectively at points A_1, B_1, and C_1. Let L_A be the line passing through A that is parallel to B_1C_1, and define lines L_B and L_C in a similar manner. Let A_2 be the intersection of L_B and L_C, and define points B_2 and C_2 in a similar manner. Let M_A, M_B, and M_C be the midpoints of sides BC, CA, and AB respectively. It is well known from the cevian nest theorem that A_2M_A, B_2M_B, and C_2M_C concur at a point which we will call P'. Given any 2 points R and S in $\triangle ABC$, prove that $RS \parallel R'S'$.



Proof: We will prove that P' is the complement of P with respect to $\triangle ABC$, which will immediately imply the desired result. Let C_1B_1 intersect BC at X. It is well known that (B, A_1, C, X) is a harmonic bundle, so taking a pencil at A shows (C_1, A_3, B_1, X) to be a harmonic bundle, and taking another pencil at B shows (A, A_3, P, A_1) to be a harmonic bundle. (1) is now evident.

(1) $\frac{\sin(\angle AC_1A_3)\sin(\angle PC_1A_1)}{\sin(\angle A_3C_1P)\sin(\angle A_1C_1B)} = 1.$

Noting that A_2M_A is a median in $\triangle AC_2B$ gives us (2).

(2) $\frac{\sin(\angle M_CAC_2)}{\sin(\angle M_CC_2A)} = \frac{\sin(\angle M_CBC_2)}{\sin(\angle M_CC_2B)} \to \frac{\sin(\angle M_CAC_2)\sin(\angle M_CC_2B)}{\sin(\angle M_CCA_2)\sin(\angle M_CBC_2)} = 1.$

Noting that $C_2B_2 \parallel C_1B_1$ and $C_2A_2 \parallel C_1A_1$ yields $\angle M_CAC_2 = \angle AC_1B_1 = \angle AC_1A_3$ and $\angle M_CBC_2 = \angle A_1C_1B$. It is now evident from (1) and (2) that $\frac{\sin(\angle AC_2M_C)}{\sin(\angle BC_2M_C)} = \frac{\sin(\angle B_1C_1P)}{\sin(\angle PC_1B_1)}$, so it can be seen that $\angle AC_2M_C = \angle B_1C_1P$ and hence $C_2M_C \parallel C_1P$. It can similarly be shown that $A_2M_A \parallel A_1P$ and $B_2M_B \parallel B_1P$. Let G be the centroid of $\triangle ABC$. Let PG intersect A_2M_A at P_A, and define P_B and P_C in a similar manner. Note that $\triangle P_C M_C G \sim \triangle PCG$, so $\frac{P_C G}{GP} = \frac{M_C G}{GC} = \frac{1}{2}$. Similarly, it can be shown that $\frac{P_A G}{GP} = \frac{1}{2}$ and $\frac{P_B G}{GP} = \frac{1}{2}$, so it is evident that $P_A = P_B = P_C = P'$, and P' is the complement of P with respect to $\triangle ABC$ as desired. ■