Problem 3.3.56.a: For what values of a, b, c, and d is the field

$$\mathbf{F} = \langle ax + by, cx + dy \rangle \text{ conservative?}$$

Solution: Writing $\mathbf{F} = \langle F_1, F_2 \rangle$, we want to pick a, b, c, and d so that $(F_1)_y = (F_2)_x$. Noting that

$$(F_1)_y = b \text{ and } (F_2)_x = c,$$

we see that $(F_1)_y = (F_2)_x$ if and only if $b = c$. So the condition on a, b, c, and d that makes $\mathbf{F}$ a conservative vector field is $\boxed{b = c}$.

Problem 3.3.56.b: For what values of a, b, and c is the field

$$\mathbf{F} = \langle ax^2 - by^2, cxy \rangle \text{ conservative?}$$

Solution: Writing $\mathbf{F} = \langle F_1, F_2 \rangle$, we want to pick a, b, and c so that $(F_1)_y = (F_2)_x$. Noting that

$$(F_1)_y = -2by \text{ and } (F_2)_x = cy$$

we see that $(F_1)_y = (F_2)_x$ if and only if $c = -2b$. So the condition on a, b, and c that makes $\mathbf{F}$ conservative is $\boxed{c = -2b}$.

(Altered) Problem 3.3.43: Consider the vector field

$$\mathbf{F} = \langle 2xy + z^2, x^2, 2xz+1 \rangle$$

and the circle C that is parameterized by

$$\mathbf{r}(t) = \langle 3\cos(t), 4\cos(t), 5\sin(t) \rangle \text{ for } 0 \le t \le 2\pi.$$

Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Solution 1: Writing $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$, we see that

$$(F_1)_y = 2x = (F_2)_x, (F_1)_z = 2z = (F_3)_x \text{ and } (F_2)_z = 0 = (F_3)_y,$$

so $\mathbf{F}$ is a conservative vector field. Since $\mathbf{F}$ is a conservative vector field and C is a closed loop, we see that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Solution 2: Alternatively, after noting that $\mathbf{F}$ is conservative, we can try to find a potential function φ for $\mathbf{F}$. This alternative procedure is only necessary if the curve C is not a closed loop, but we will do it here anyways just for the additional practice. Following standard procedure, we see that

$$\varphi(x,y,z) = \int F_1(x,y,z)dx + h(y,z) = \int (2xy + z^2)dx + h(y,z)$$

$$= x^2y + xz^2 + h(y,z) \longrightarrow x^2 = F_2(x,y,z) = \frac{\partial}{\partial y}\varphi(x,y,z) = x^2 + h_y(y,z)$$

$$\longrightarrow h_y(y,z) = 0 \longrightarrow h(y,z) = g(z) \longrightarrow \varphi(x,y,z) = x^2y + xz^2 + g(z)$$

$$\longrightarrow 2xz + 1 = F_3(x,y,z) = \frac{\partial}{\partial z}\varphi(x,y,z) = 2xz + g_z(z)$$

$$\longrightarrow g_z(z) = 1 \longrightarrow g(z) = z + c \longrightarrow \varphi(x, y, z) = x^2y + xz^2 + z + c.$$

Since we only need to pick any 1 of the many possible potential functions, let us set $c = 0$ and work with

$$\varphi(x, y, z) = x^2y + xz^2 + z.$$

We are now in position to use the Fundamental Theorem for Line Integrals to see that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{r}(2\pi)) - \varphi(\mathbf{r}(0)) = \varphi(3, 4, 0) - \varphi(3, 4, 0) = \boxed{0}.$$

"Solution" 3: We note that

$$\mathbf{r}'(t) = \langle -3\sin(t), -4\sin(t), 5\cos(t)\rangle, \text{ so}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{2\pi} \langle 2(3\cos(t))(4\cos(t)) + (5\sin(t))^2, (3\cos(t))^2, 2(3\cos(t))(5\sin(t)) + 1\rangle \cdot \langle -3\sin(t), -4\sin(t), 5\cos(t)\rangle dt$$

Then you give up because there is a lot of algebra and the resulting integral is pretty hard to do.

Problem 3.2.34: Consider the vector field $\mathbf{F} = \langle -y, x \rangle$ and the semicircle C that is parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t) \rangle$, for $0 \leq t \leq \pi$. Evaluate

$$\int_C \mathbf{F} \cdot \mathbf{T} ds.$$

Solution: Noting that

$$\mathbf{T} ds = d\mathbf{r} = \langle -4\sin(t), 4\cos(t) \rangle dt,$$

We see that

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}$$
$$= \int_0^\pi \langle -4\sin(t), 4\cos(t) \rangle \cdot \langle -4\sin(t), 4\cos(t) \rangle dt$$
$$= \int_0^\pi (16\sin^2(t) + 16\cos^2(t)) dt = \int_0^\pi 16 dt = 16t\Big|_{t=0}^{\pi} = \boxed{16\pi}.$$

Remark: We note that $\mathbf{F}$ is not a conservative vector field, so we could not use the Fundamental Theorem for Line Integrals.

Problem 3.2.28: Let C be the line segment between $(1,4,1)$ and $(3,6,3)$. Evaluate the scalar line integral

$$\int_C \frac{xy}{z} ds.$$

Solution: First, we parameterize the curve C using the standard procedure for parameterizing a line segment. We see that

$$\mathbf{r}(t) = \langle 1,4,1\rangle + t\left(\langle 3,6,3\rangle - \langle 1,4,1\rangle\right) = \langle 1,4,1\rangle + \langle 2t,2t,2t\rangle$$

$$= \langle 1+2t, 4+2t, 1+2t\rangle \text{ for } 0 \leq t \leq 1.$$

Recalling that $ds = |\mathbf{r}'(t)|dt$, we see that

$$\mathbf{r}'(t) = \langle 2,2,2\rangle \text{ so,}$$

$$ds = |\langle 2,2,2\rangle|dt = \sqrt{2^2+2^2+2^2}dt = \sqrt{12}dt = 2\sqrt{3}dt.$$

Putting everything together, we see that

$$\int_C \frac{xy}{z} ds = \int_0^1 \frac{(1+2t)(4+2t)}{(1+2t)} 2\sqrt{3}dt = \int_0^1 (4+2t)2\sqrt{3}dt = \int_0^1 (8\sqrt{4}+4\sqrt{3}t)dt$$

$$= 8\sqrt{3}t + 2\sqrt{3}t^2\Big|_{t=0}^{1} = \boxed{10\sqrt{3}}.$$

Problem 4.6.48: Let A be the 2×2 matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Choose some vector $\mathbf{b} \in \mathbb{R}^2$ for which the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent. Then verify that the associated equation $A^T A\mathbf{x} = A^T\mathbf{b}$ is consistent for your choice of $\mathbf{b}$. Let $\mathbf{x}^*$ be a solution to $A^T A\mathbf{x} = A^T\mathbf{b}$ and let $\mathbf{x} \in \mathbb{R}^2$ be random. Verify that $||A\mathbf{x}^* - \mathbf{b}|| \leq ||A\mathbf{x} - \mathbf{b}||$.

Solution: We see that if

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

then the equation $A\mathbf{x} = \mathbf{b}$ is represented by the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 6 & b_2 \end{array}\right].$$

By row reducing, we see that

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 6 & b_2 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 3b_1 \end{array}\right].$$

It follows that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if $b_2 - 3b_1 \neq 0$, so we may take

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix},$$

we see that the equation $A^T A\mathbf{x} = A^T\mathbf{b}$ becomes

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix},$$

which is represented by the augmented matrix

$$\left[\begin{array}{cc|c} 10 & 20 & 4 \\ 20 & 40 & 8 \end{array}\right].$$

By row reducing, we see that

$$\left[\begin{array}{cc|c} 10 & 20 & 4 \\ 20 & 40 & 8 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 10 & 20 & 4 \\ 0 & 0 & 0 \end{array}\right],$$

which shows us that

$$10x_1 + 20x_2 = 4 \longrightarrow x_1 = \frac{2}{5} - 2x_2.$$

It follows that the general solution to $A^T A\mathbf{x} = A^T\mathbf{b}$ is given by

$$\mathbf{x} = \begin{bmatrix} \frac{2}{5} \\ 0 \end{bmatrix} + x \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Letting $x = 0$, we see that we can take

$$\mathbf{x}^* = \begin{bmatrix} \frac{2}{5} \\ 0 \end{bmatrix}$$

as a solution to $A^T A\mathbf{x} = A^T\mathbf{b}$. We may take

$$\mathbf{x} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

as our random $\mathbf{x} \in \mathbb{R}^2$. We now see that

$$||A\mathbf{x}^* - \mathbf{b}|| = || \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} \frac{2}{5} \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} || = || \begin{bmatrix} \frac{2}{5} \\ \frac{6}{5} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} || = || \begin{bmatrix} -\frac{3}{5} \\ \frac{1}{5} \end{bmatrix} ||$$

$$= \sqrt{(-\frac{3}{5})^2 + (\frac{1}{5})^2} = \sqrt{\frac{10}{25}} = \frac{\sqrt{10}}{5}$$

and

$$||A\mathbf{x}-\mathbf{b}|| = || \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} || = || \begin{bmatrix} 3 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} || = || \begin{bmatrix} 2 \\ 8 \end{bmatrix} ||$$

$$= \sqrt{(2)^2 + (8)^2} = \sqrt{68} = 2\sqrt{17} \geq \frac{\sqrt{10}}{5},$$

so

$$||A\mathbf{x}^* - \mathbf{b}|| \leq ||A\mathbf{x} - \mathbf{b}||$$

as claimed.

Problem 3.3.41: Evaluate

$$\int_C \triangle(e^{-x}\cos(y)) \cdot d\mathbf{r},$$

where C is the line segment from $(0,0)$ to $(\ln(2), 2\pi)$.

Solution: $\triangle(e^{-x}\cos(y))$ is a conservative vector field with potential function $\varphi(x,y) = e^{-x}\cos(y)$, so by the Fundamental Theorem for Line Integrals, we see that

$$\int_C \triangle(e^{-x}\cos(y)) \cdot d\mathbf{r} = \varphi(\ln(2), 2\pi) - \varphi(0,0) = e^{\ln(2)}\cos(2\pi) - e^0\cos(0)$$

$$= 2 - 1 = \boxed{1}.$$

Problem 4.5.45: Find the general solution to the system of linear equations represented by the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

and express it in vector form.

Solution: We see that if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

is a solution, then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\to x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - x_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - x_5 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + x_5 \\ -2x_3 - x_5 \\ -x_5 \end{bmatrix}$$

so $x_1 = x_3 + x_5$, $x_2 = -2x_3 - x_5$, and $x_4 = -x_5$. We now see that the general solution is given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \boxed{x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \text{ where } x_3 \text{ and } x_5 \text{ are free.}}$$

Problem 4.7.51: Given a linearly independent set of vectors $\{v_1, v_2, v_3\} \subseteq \mathbb{R}^m$, show that the set of vectors $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also linearly independent.

Solution: Let $c_1, c_2, c_3 \in \mathbb{R}$ be such that

$$0 = c_1v_1 + c_2(v_1 + v_2) + c_3(v_1 + v_2 + v_3) = c_1v_1 + c_2v_1 + c_2v_2 + c_3v_1 + c_3v_2 + c_3v_3$$

$$= (c_1 + c_2 + c_3)v_1 + (c_2 + c_3)v_2 + c_3v_3.$$

Since $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors, we see that

$$\begin{array}{rrrcl} c_1+ & c_2+ & c_3 & = & 0 \\ & c_2+ & c_3 & = & 0 \\ & & c_3 & = & 0 \end{array} \longrightarrow \begin{array}{rrrcl} c_1 & & & = & 0 \\ & c_2 & & = & 0 \\ & & c_3 & = & 0 \end{array} \tag{0.1}$$

Since (c_1, c_2, c_3) must be $(0, 0, 0)$, we see that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is indeed a linearly independent set of vectors.

Problem 18 (from the chapter 4 Review): Given

$$A^{-1} = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 2 & 1 \\ 4 & -4 & 3 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} -6 & 4 & 3 \\ 7 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix}$$

evaluate

$$[(A^{-1}B^{-1})^{-1}A^{-1}B]^{-1}$$

Solution: We see that

$$[(A^{-1}B^{-1})^{-1}A^{-1}B]^{-1} = [(B^{-1})^{-1}(A^{-1})^{-1}A^{-1}B]^{-1} = [BAA^{-1}B]^{-1}$$

$$[BB]^{-1} = B^{-1}B^{-1}$$

$$= \begin{bmatrix} -6 & 4 & 3 \\ 7 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -6 & 4 & 3 \\ 7 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 70 & -19 & 5 \\ -39 & 44 & 21 \\ 11 & 8 & 22 \end{bmatrix}$$