Problem 6.2.27 (Not part of the final this year): Consider the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{1}$$

Show that for a solution $u(r,\theta) = R(r)\Theta(\theta)$ having separated variables, we must have

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0, \text{ and} \tag{2}$$

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \tag{3}$$

where λ is some constant.

Solution: We begin by plugging $u(r,\theta) = R(r)\Theta(\theta)$ into equation (1) to see that

$$0 = \frac{\partial^2}{\partial r^2}(R(r)\Theta(\theta)) + \frac{1}{r}\frac{\partial}{\partial r}(R(r)\Theta(\theta)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R(r)\Theta(\theta)) \tag{4}$$

$$= R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) \tag{5}$$

$$\to -\frac{1}{r^2}R(r)\Theta''(\theta) = R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) \tag{6}$$

$$\to \frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{R''(r) + \frac{1}{r}R'(r)}{-\frac{1}{r^2}R(r)} \overset{*}{=} \gamma. \tag{7}$$

To derive equation (2) we use equation (7) to see that

$$\frac{R''(r) + \frac{1}{r}R'(r)}{-\frac{1}{r^2}R(r)} = \gamma \to R''(r) + \frac{1}{r}R(r) = -\frac{\gamma}{r^2}R(r) \tag{8}$$

$$\to R''(r) + \frac{1}{r}R'(r) + \frac{\gamma}{r^2}R(r) = 0 \to r^2R''(r) + rR'(r) + \gamma R(r) = 0. \tag{9}$$

To derive equation (3) we use equation (7) to see that

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = \gamma \to \Theta''(\theta) = \gamma\Theta(\theta) \to \Theta''(\theta) - \gamma\Theta(\theta) = 0. \tag{10}$$

We now see that we can pick our constant λ as $\lambda = -\gamma$.

Problem 6.2.30 (Not part of the final this year): Consider the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{11}$$

Show that for a solution $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ having separated variables, we must have

$$\Theta''(\theta) + \mu\Theta(\theta) = 0, \tag{12}$$

$$Z''(z) + \lambda Z(z) = 0, \text{ and} \tag{13}$$

$$r^2R''(r) + rR'(r) - (r^2\lambda + \mu)R(r) = 0, \tag{14}$$

where μ and λ are constants.

Solution: We proceed as in problem 6.2.27 and plug $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ into equation (11) to see that

$$\frac{\partial^2}{\partial r^2}(R(r)\Theta(\theta)Z(z)) + \frac{1}{r}\frac{\partial}{\partial r}(R(r)\Theta(\theta)Z(z)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R(r)\Theta(\theta)Z(z)) + \frac{\partial^2}{\partial z^2}(R(r)\Theta(\theta)Z(z)) = 0 \tag{15}$$

$$\to R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) + R(r)\Theta(\theta)Z''(z) = 0. \tag{16}$$

We will now try to derive equation (13) from equation (16). Beginning with equation (16) we see that

$$R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) + R(r)\Theta(\theta)Z''(z) = 0. \tag{17}$$

$$-R(r)\Theta(\theta)Z''(z) = R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) \tag{18}$$

$$\to \frac{Z''(z)}{Z(z)} = \frac{R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta)}{-R(r)\Theta(\theta)} \overset{*}{=} -\lambda \tag{19}$$

$$\to Z''(z) = -\lambda Z(z) \to Z''(z) + \lambda Z(z) = 0. \tag{20}$$

We will now derive equation (12) from equation (16). Beginning with equation (16) we see that

$$R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) + R(r)\Theta(\theta)Z''(z) = 0. \tag{21}$$

$$-\frac{1}{r^2}R(r)\Theta''(\theta)Z(z) = R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + R(r)\Theta(\theta)Z''(z) \tag{22}$$

$$\to \frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z''(z)}{-\frac{1}{r^2}R(r)Z(z)} \stackrel{*}{=} -\mu \tag{23}$$

$$\to \Theta''(\theta) = -\mu\Theta(\theta) \to \Theta''(\theta) + \mu\Theta(\theta) = 0. \tag{24}$$

Lastly, we will derive equation (14) from equation (16). Beginning with equation (16) we see that

$$R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) + R(r)\Theta(\theta)Z''(z) = 0. \tag{25}$$

$$R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) = -\frac{1}{r^2}R(r)\Theta''(\theta)Z(z) - R(r)\Theta(\theta)Z''(z) \tag{26}$$

$$\to \frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} = \frac{-\frac{1}{r^2}\Theta''(\theta)Z(z) - \Theta(\theta)Z''(z)}{\Theta(\theta)Z(z)} = -\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{-Z''(z)}{Z(z)} = \frac{\mu}{r^2} + \lambda \tag{27}$$

$$\to R''(r) + \frac{1}{r}R'(r) = (\frac{\mu}{r^2} + \lambda)R(r) \to R''(r) + \frac{1}{r}R'(r) - (\frac{\mu}{r^2} + \lambda)R(r) = 0 \tag{28}$$

$$\to r^2R''(r) + rR'(r) - (\mu + r^2\lambda)R(r) = 0. \tag{29}$$

Problem 6.2.13: Find the values of λ (eigenvalues) for which the following problem has a nontrivial solution. Also determine the corresponding nontrivial solutions (eigenfunctions).

$$y'' + \lambda y = 0; \quad 0 < x < \pi, \quad y(0) - y'(0) = 0,\ y(\pi) = 0. \tag{30}$$

Solution: We begin by examining the characteristic equation for equation (30) and see that

$$r^2 + \lambda = 0 \to r = \pm\sqrt{-\lambda}. \tag{31}$$

We now consider 3 separate cases based on the sign of λ.

Case 1: $\lambda = 0$.

In this case we see that $r = 0$ is a double root of the characteristic equation, so the general solution to equation (30) is

$$y(t) = c_1 e^{0\cdot t} + c_2 t e^{0\cdot t} = c_1 + c_2 t. \tag{32}$$

Noting that

$$y'(t) = c_2, \tag{33}$$

we proceed to make use of the initial conditions to see that

$$\begin{array}{rcl} 0 & = & y(0) - y'(0) & = & c_1 - c_2 \\ 0 & = & y(\pi) & = & c_1 + \pi c_2 \end{array} \to \begin{array}{rcl} c_1 & = & c_2 \\ c_1 & = & -\pi c_2 \end{array} \to (c_1, c_2) = (0,0), \tag{34}$$

so we only have trivial solutions in this case.

Case 2: $\lambda < 0$.

In this case we see that $r = \sqrt{-\lambda}$ and $r = -\sqrt{-\lambda}$ are distinct real roots of the characteristic equation, so the general solution to equation (30) is

$$y(t) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}. \tag{35}$$

Noting that

$$y'(t) = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}t} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}t}, \tag{36}$$

we proceed to make use of the initial conditions to see that

$$\begin{array}{rclcl} 0 & = & y(0) - y'(0) & = & c_1(1-\sqrt{-\lambda}) + c_2(1+\sqrt{-\lambda}) \\ 0 & = & y(\pi) & = & c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} \end{array} \tag{37}$$

$$\to \underbrace{\begin{bmatrix} 1-\sqrt{-\lambda} & 1+\sqrt{-\lambda} \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \text{ Since} \tag{38}$$

$$\det(A) = e^{-\sqrt{-\lambda}\pi}(1-\sqrt{-\lambda}) - e^{\sqrt{-\lambda}\pi}(1+\sqrt{-\lambda}) < 0, \tag{39}$$

we see that $\det(A) \neq 0$, so A is a nonsingular matrix. It follows that equation (38) only has the trivial solution of $(c_1, c_2) = (0, 0)$, so we only have trivial solutions to equation (30) in this case as well.

Case 3: $\lambda > 0$.

In this case we see that $r = \sqrt{-\lambda}$ and $r = -\sqrt{-\lambda}$ are distinct complex roots of the characteristic equation, so the general solution to equation (30) is

$$y(t) = c_1' e^{\sqrt{-\lambda}t} + c_2' e^{-\sqrt{-\lambda}t} = c_1\cos(\sqrt{\lambda}t) + c_2\sin(\sqrt{\lambda}t). \tag{40}$$

Noting that

$$y'(t) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}t) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}t), \tag{41}$$

we proceed to make use of the initial conditions to see that

$$\begin{aligned} 0 &= y(0) - y'(0) &&= c_1 - c_2\sqrt{\lambda} \\ 0 &= y(\pi) &&= c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) \end{aligned} \tag{42}$$

$$\rightarrow \begin{aligned} c_1 &= c_2\sqrt{\lambda} \\ 0 &= c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) \end{aligned} \tag{43}$$

$$\rightarrow \begin{aligned} c_1 &= c_2\sqrt{\lambda} \\ 0 &= c_2\left(\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi)\right) \end{aligned}. \tag{44}$$

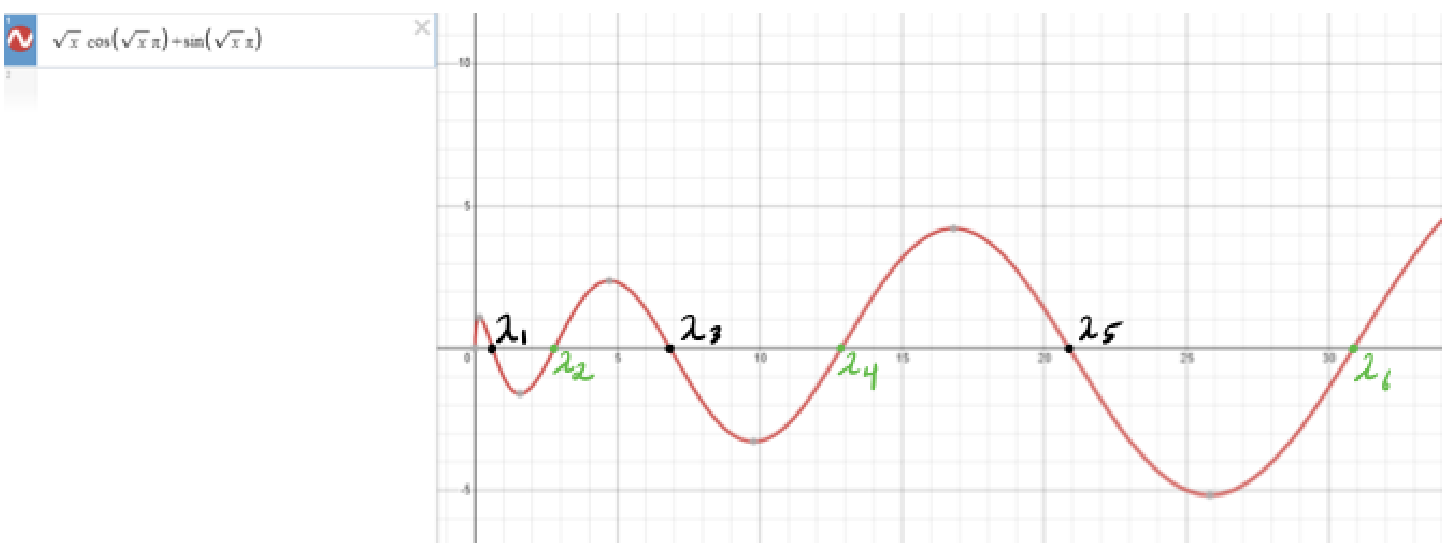
In order to have nontrivial solutions to equation (30) we need to have nontrivial solutions to system of equations in (44). We see that $c_1 = 0$ if and only if $c_2 = 0$, and that c_2 will be 0 if

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) \neq 0. \tag{45}$$

It follows that we want to find the values of λ for which

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) = 0, \tag{46}$$

so that we can find a corresponding $c_2 \neq 0$. Sadly, equation (46) is not something that can be explicitly solved by hand. Therefore, we let $\{\lambda_n\}_{n=1}^{\infty}$ denote the solutions to equation (46) as shown in the picture below.



To be precise, we know that the solutions to equation (46) exist even though we cannot write down exactly what they are, so we talk about them by enumerating them as $\{\lambda_n\}_{n=1}^{\infty}$.

We note that for any $n \geq 1$, if $\lambda = \lambda_n$, then the second equation in (44) holds for any value of c_2, so we will have $(c_1, c_2) = (c_2\sqrt{\lambda_n}, c_2)$ is a nontrivial solution to equation (30). In conclusion, the eigenvalues of (30) are $\{\lambda_n\}_{n=1}^{\infty}$ and the eigen functions corresponding to any given λ_n are

$$y(t) = c\left(\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}t) + \sin(\sqrt{\lambda_n}t)\right); c \in \mathbb{R}. \tag{47}$$

Problem 6.3.11: Find the fourier series of the function

$$f(x) = \begin{cases} 1 & \text{if } -2 < x < 0 \\ x & \text{if } 0 < x < 2 \end{cases}, \tag{48}$$

over the interval $[-2, 2]$.

Solution: Since our interval has a radius of $L = 2$, we see that the basis we will work with is $(\sin(\frac{2\pi nx}{2L}))_{n=1}^{\infty} \cup (\cos(\frac{2\pi mx}{2L}))_{m=1}^{\infty}$ which simplifies to $(\sin(\frac{\pi nx}{2}))_{n=1}^{\infty} \cup (\cos(\frac{\pi mx}{2}))_{m=1}^{\infty}$. We may now let a_0, $(a_n)_{n=1}^{\infty}$, and $(b_n)_{n=1}^{\infty}$ be such that

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi nx}{2}) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi nx}{2}). \tag{49}$$

First let us determine the sequence $(b_n)_{n=1}^{\infty}$. We note that for each $n \geq 1$ we have

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{2\pi nx}{2L})dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{\pi nx}{2})dx \tag{50}$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{\pi n x}{2})dx = \frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi n x}{2})dx + \frac{1}{2}\int_{0}^{2} x\sin(\frac{\pi n x}{2})dx. \tag{51}$$

We see that

$$\frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi n x}{2})dx = -\frac{1}{\pi n}\cos(\frac{\pi n x}{2})\Big|_{x=-2}^{0} = -\frac{2}{\pi n} + \frac{2}{\pi n}\cos(-\pi n) \tag{52}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\frac{2}{\pi n} & \text{if n is odd} \end{cases}. \tag{53}$$

Using integration by parts, we also see that

$$\frac{1}{2}\int_{0}^{2} x\sin(\frac{\pi n x}{2})dx = -\frac{1}{\pi n}x\cos(\frac{\pi n x}{2})\Big|_{x=0}^{2} - \int_{0}^{2} -\frac{2}{\pi n}\cos(\frac{\pi n x}{2})dx \tag{54}$$

$$= -\frac{2}{\pi n}\cos(\pi n) + \left(\frac{2}{\pi^2 n^2}\sin(\frac{\pi n x}{2})\Big|_{x=0}^{2}\right) = -\frac{2}{\pi n}\cos(\pi n) \tag{55}$$

$$= \begin{cases} -\frac{2}{\pi n} & \text{if n is even} \\ \frac{2}{\pi n} & \text{if n is odd} \end{cases}. \tag{56}$$

Putting all of this together, we see that for $n \geq 1$ we have

$$b_n = \frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi n x}{2})dx + \frac{1}{2}\int_{0}^{2} x\sin(\frac{\pi n x}{2})dx = \begin{cases} -\frac{2}{\pi n} & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}. \tag{57}$$

Now let us determine the sequence $(a_n)_{n=1}^{\infty}$. We note that for $n \geq 1$ we have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{2\pi n x}{2L})dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{\pi n x}{2})dx \tag{58}$$

$$= \frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi n x}{2})dx + \frac{1}{2}\int_{0}^{2} x\cos(\frac{\pi n x}{2})dx. \tag{59}$$

We see that

$$\frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi n x}{2})dx = \frac{1}{\pi n}\sin(\frac{\pi n x}{2})\Big|_{x=-2}^{0} = 0. \tag{60}$$

Using integration by parts, we also see that

$$\frac{1}{2}\int_{0}^{2}x\cos(\frac{\pi n x}{2})dx = \frac{1}{\pi n}x\sin(\frac{\pi n x}{2})\Big|_{x=0}^{2} - \int_{0}^{2}\frac{2}{\pi n}\sin(\frac{\pi n}{2})dx \tag{61}$$

$$= -\frac{1}{\pi n}\int_{0}^{2}\sin(\frac{\pi n x}{2})dx = \frac{2}{\pi^2 n^2}\cos(\frac{\pi n x}{2})\Big|_{x=0}^{2} \tag{62}$$

$$= \frac{2}{\pi^2 n^2}(\cos(\pi n) - 1) = \begin{cases} 0 & \text{if n is even} \\ \frac{-4}{\pi^2 n^2} & \text{if n is odd} \end{cases}. \tag{63}$$

Putting all of this together, we see that for $n \geq 1$ we have

$$a_n = \frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi n x}{2})dx + \frac{1}{2}\int_{0}^{2}x\cos(\frac{\pi n x}{2})dx = \begin{cases} 0 & \text{if n is even} \\ -\frac{4}{\pi^2 n^2} & \text{if n is odd} \end{cases}. \tag{64}$$

Lastly, we see that

$$a_0 = \frac{1}{2L}\int_{-L}^{L}f(x)dx = \frac{1}{4}\int_{-2}^{2}f(x)dx = \frac{1}{4}\int_{-2}^{0}1dx + \frac{1}{4}\int_{0}^{2}xdx \tag{65}$$

$$\frac{1}{2} + \left(\frac{x^2}{8}\Big|_{x=0}^{2}\right) = 1. \tag{66}$$

Finally, we see that

$$f(x) \sim 1 + \left(\sum_{n=1}^{\infty}\frac{2}{\pi^2 n^2}((-1)^n - 1)\cos(\frac{\pi n}{2}x)\right) + \left(\sum_{n=1}^{\infty}\frac{1}{\pi n}((-1)^{n+1} - 1)\sin(\frac{\pi n x}{2})\right) \tag{67}$$

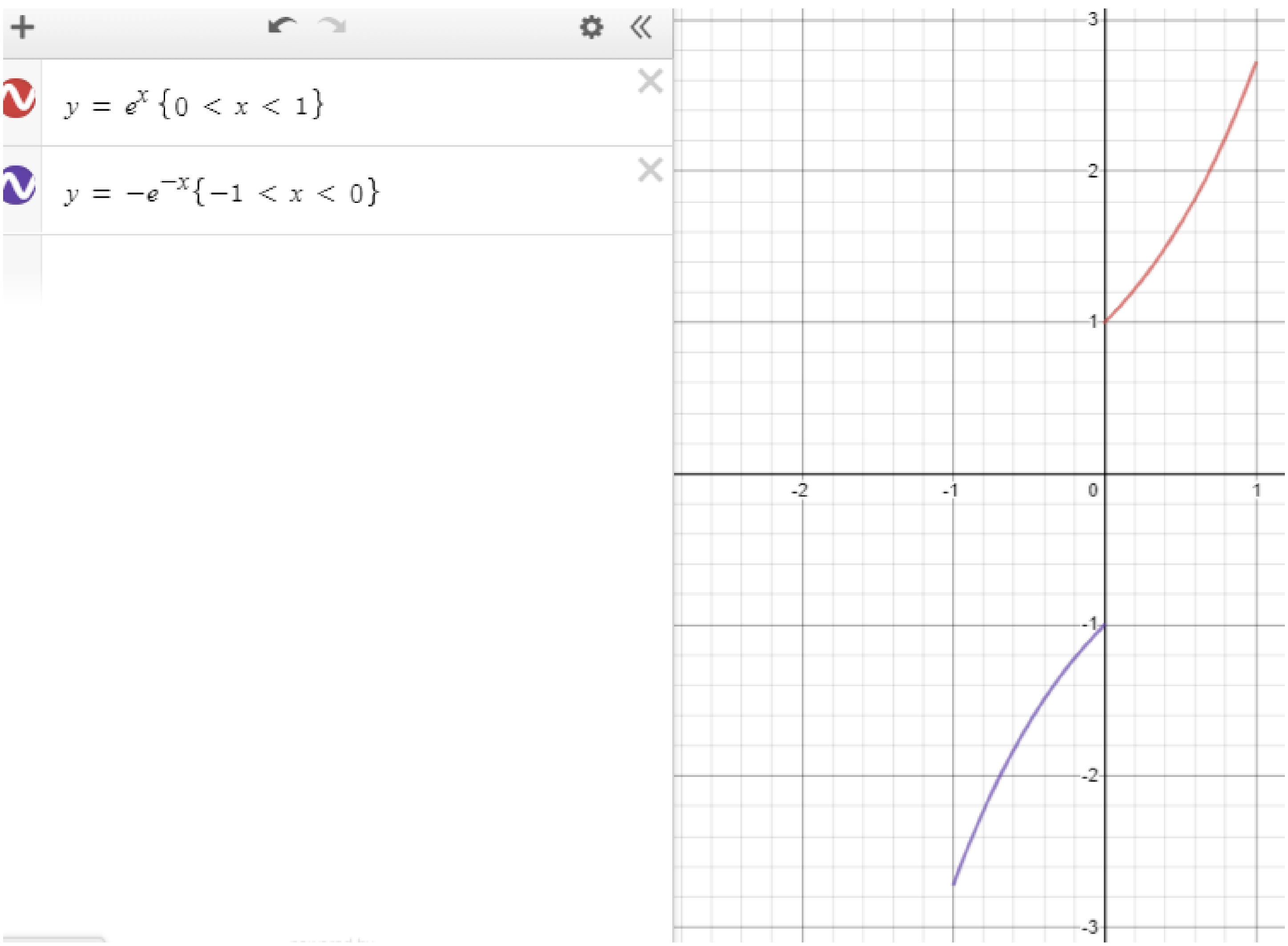
Problem 6.4.10: Find the Fourier sine series for

$$f(x) = e^x, \quad 0 < x < 1. \tag{68}$$

Solution: The fourier sine series of $f(x)$ is just the fourier series of $g(x)$, the odd 2-periodic extension of $f(x)$, which is the 2-periodic function defined by the formula

$$g(x) = \begin{cases} f(x) & \text{if } \quad 0 < x < 1 \\ -f(-x) & \text{if } -1 < x < 0 \end{cases}. \tag{69}$$

Below is a graph of $g(x)$ restricted to the interval $(-1, 1)$. The red portion of the graph is also the graph of $f(x)$.



Since $g(x)$ is an odd function (by construction, this will always be the case) the fourier series of $g(x)$ will not have any cosine terms in it. We see that for any $n \geq 1$, we have

$$b_n = \frac{1}{1}\int_{-1}^{1} g(x)\sin\left(\frac{2n\pi x}{2}\right)dx \overset{\text{by oddness}}{=} \frac{2}{1}\int_{0}^{1} f(x)\sin(n\pi x)dx \tag{70}$$

$$= 2\int_0^1 e^x \sin(n\pi x)dx = 2\int_0^1 \frac{e^{(1+n\pi i)x} - e^{(1-n\pi i)x}}{2i}dx \tag{71}$$

$$= -i\int_0^1 (e^{(1+n\pi i)x} - e^{(1-n\pi i)x})dx = -i\left(\frac{e^{(1+n\pi i)x}}{1+n\pi i} - \frac{e^{(1-n\pi i)x}}{1-n\pi i}\right)\Big|_0^1 \tag{72}$$

$$= \left(\frac{e^{1+n\pi i}}{1+n\pi i} - \frac{e^{1-n\pi i}}{1-n\pi i}\right) - \left(\frac{e^0}{1+n\pi i} - \frac{e^0}{1-n\pi i}\right) \tag{73}$$

$$= \left(\frac{e(\cos(n\pi) + i\sin(n\pi))}{1+n\pi i} - \frac{e(\cos(n\pi) + i\sin(-n\pi))}{1-n\pi i}\right) - \left(\frac{1}{1+n\pi i} - \frac{1}{1-n\pi i}\right) \tag{74}$$

$$= \frac{e(-1)^n - 1}{1+n\pi i} - \frac{e(-1)^n - 1}{1-n\pi i} = \frac{2e(-1)^n - 2}{1+n^2\pi^2} \tag{75}$$

$$\to \boxed{f(x) \sim \sum_{n=1}^{\infty} \frac{2e(-1)^n - 2}{1+n^2\pi^2}\sin(n\pi x)}. \tag{76}$$

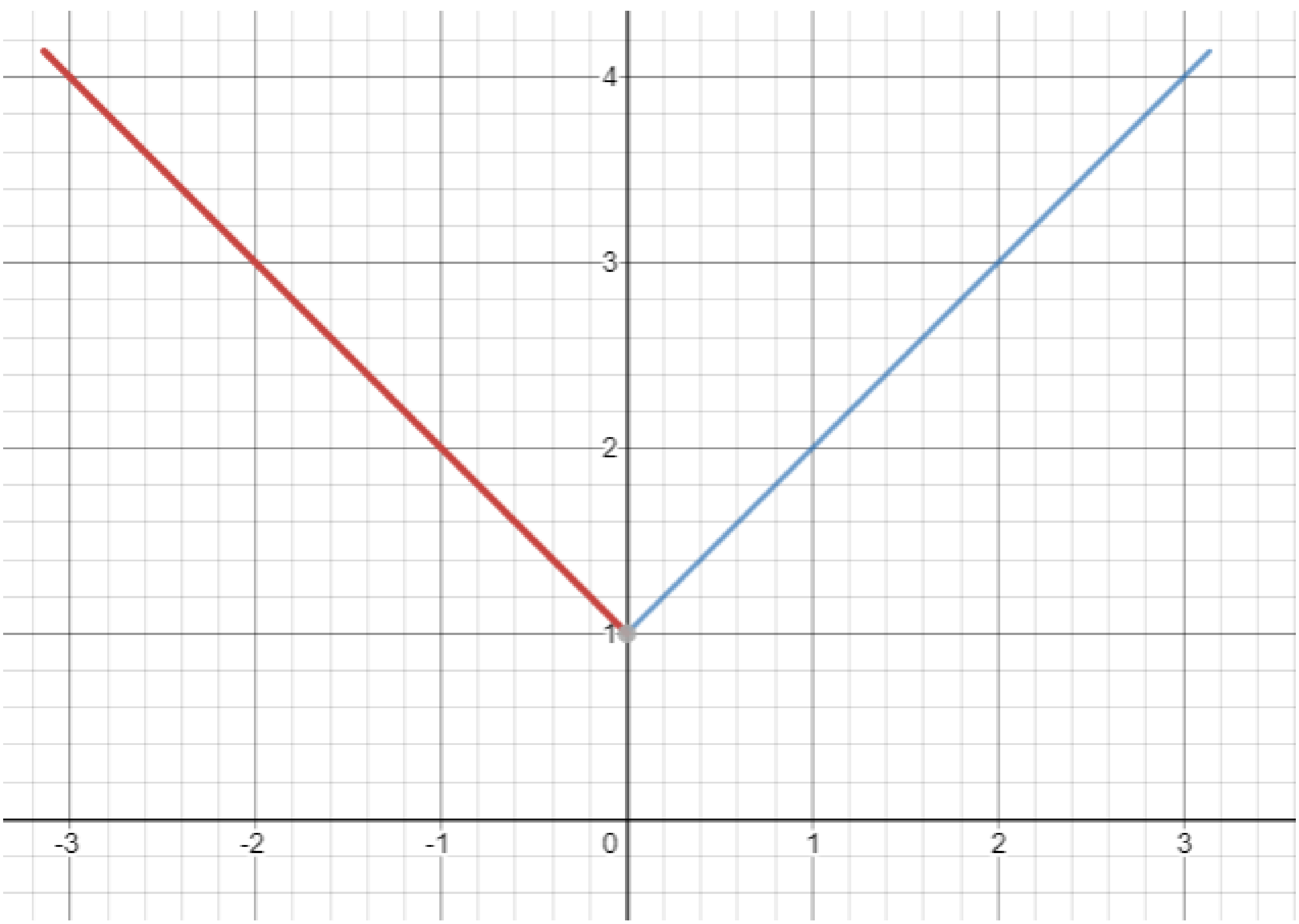
Problem 6.4.12: Find the Fourier cosine series for

$$f(x) = 1 + x, \quad 0 < x < \pi. \tag{77}$$

Solution: The fourier cosine series of $f(x)$ is just the fourier series of $g(x)$, the even 2π-periodic extension of $f(x)$, which is the 2π-periodic function defined by the formula

$$g(x) = \begin{cases} f(x) & \text{if} \quad 0 < x < \pi \\ f(-x) & \text{if} \ -\pi < x < 0 \end{cases}. \tag{78}$$

Below is a graph of $g(x)$ restricted to the interval $(-\pi, \pi)$. The blue portion of the graph is also the graph of $f(x)$.



Since $g(x)$ is an even function (by construction, this will always be the case) the fourier series of $g(x)$ will not have any sine terms in it. We see that for any $n \geq 1$, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(\frac{2\pi n x}{2\pi})dx \overset{\text{by evenness}}{=} \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \tag{79}$$

$$= \frac{2}{\pi}\int_0^{\pi}(1+x)\cos(nx)dx = \frac{2}{\pi}\cdot(1+x)\frac{\sin(nx)}{n}\Big|_{x=0}^{\pi} - \frac{2}{\pi}\int_0^{\pi} 1\cdot\frac{\sin(nx)}{n}dx \tag{80}$$

$$= 0 - \frac{2}{\pi}\left(\frac{-\cos(nx)}{n^2}\Big|_{x=0}^{\pi}\right) = \frac{2\cos(n\pi)-2}{\pi n^2} = \begin{cases} 0 & \text{if n is even} \\ \frac{-4}{\pi n^2} & \text{if n is odd} \end{cases}. \tag{81}$$

Similarly, we see that

$$a_0 \overset{*}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(1+x)dx \tag{82}$$

$$\frac{(1+x)^2}{2\pi}\Big|_{x=0}^{\pi} = \frac{(\pi+1)^2-1}{2\pi} = \frac{\pi}{2}+1. \tag{83}$$

Putting everything together, we see that

$$\boxed{f(x) \sim \frac{\pi}{2}+1+\sum_{n=0}^{\infty} -\frac{4}{\pi(2n+1)^2}\cos((2n+1)x)}. \tag{84}$$