Problem 5.2.17: Solve the following initial value problem.

$$y'' - 3y' - 18y = 0; \quad y(0) = 0, y'(0) = 4. \tag{1}$$

Solution: We see that the characteristic polynomial of equation (1) is

$$0 = r^2 - 3r - 18 = (r-6)(r+3), \tag{2}$$

which has roots $r = -3, 6$. It follows that the general solutions to equation (1) is

$$y(t) = c_1 e^{-3t} + c_2 e^{6t}. \tag{3}$$

Using the initial conditions, we see that

$$\begin{array}{rcrcrcr} 0 & = & y(0) & = & c_1 e^{-3\cdot 0} + c_2 e^{6\cdot 0} & = & c_1 + c_2 \\ 4 & = & y'(0) & = & -3c_1 e^{3\cdot 0} + 6c_2 e^{6\cdot 0} & = & -3c_1 + 6c_2 \end{array} \tag{4}$$

$$\rightarrow \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ -3c_1 & + & 6c_2 & = & 4 \end{array} \xrightarrow{R_2+3R_1} \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ & & 9c_2 & = & 4 \end{array} \tag{5}$$

$$\xrightarrow{\frac{1}{9}R_2} \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ & & c_2 & = & \frac{4}{9} \end{array} \xrightarrow{R_1-R_2} \begin{array}{rcl} c_1 & = & -\frac{4}{9} \\ c_2 & = & \frac{4}{9} \end{array} \tag{6}$$

$$\rightarrow \boxed{y(t) = -\frac{4}{9}e^{-3t} + \frac{4}{9}e^{6t}}. \tag{7}$$

Problem 5.2.23: Solve the following initial value problem.

$$y'' - y' + \frac{1}{4}y = 0; \quad y(0) = 1, y'(0) = 2. \tag{8}$$

Solution: We see that the characteristic polynomial of equation (8) is

$$0 = r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2, \tag{9}$$

which has $r = \frac{1}{2}$ as a double root. It follows that the general solutions to equation (8) is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}. \tag{10}$$

Noting that

$$y'(t) = \frac{1}{2}c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}} = (\frac{1}{2}c_1 + c_2)e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}}, \tag{11}$$

we can use the initial conditions, to see that

$$\begin{array}{ccccccc} 1 & = & y(0) & = & c_1 e^{\frac{0}{2}} + c_2 \cdot 0 \cdot e^{\frac{0}{2}} & = & c_1 \\ 2 & = & y'(0) & = & (\frac{1}{2}c_1 + c_2)e^{\frac{0}{2}} + \frac{1}{2}c_2 \cdot 0 \cdot e^{\frac{0}{2}} & = & \frac{1}{2}c_1 + c_2 \end{array} \tag{12}$$

$$\rightarrow \begin{array}{ccccc} c_1 & & & = & 1 \\ \frac{1}{2}c_1 & + & c_2 & = & 2 \end{array} \rightarrow \begin{array}{ccccc} c_1 & = & & & 1 \\ c_2 & = & 2 - \frac{1}{2} \cdot 1 & = & \frac{3}{2} \end{array} \tag{13}$$

$$\rightarrow \boxed{y(t) = e^{\frac{t}{2}} + \frac{3}{2}te^{\frac{t}{2}}}. \tag{14}$$

Problem 5.2.31: Solve the following initial value problem.

$$y'' + 6y' + 10y = 0; \quad y(0) = 0, y'(0) = 6. \tag{15}$$

Solution: We see that the characteristic polynomial of equation (15) is

$$0 = r^2 + 6r + 10 \to r = \frac{-6 \pm \sqrt{6^2 - 4\cdot 1\cdot 10}}{2\cdot 1} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i, \tag{16}$$

It follows that the general solutions to equation (15) is

$$y(t) = c_1' e^{(-3+i)t} + c_2' e^{(-3-i)t} = c_1 \sin(t)e^{-3t} + c_2 \cos(t)e^{-3t}. \tag{17}$$

Noting that

$$y'(t) = c_1\cos(t)e^{-3t} - 3c_1\sin(t)e^{-3t} - c_2\sin(t)e^{-3t} - 3c_2\cos(t)e^{-3t} \tag{18}$$

$$= (-3c_1 - c_2)\sin(t)e^{-3t} + (c_1 - 3c_2)\cos(t)e^{-3t}, \tag{19}$$

we can use the initial conditions to see that

$$\begin{array}{rcl} 0 &=& y(0) &=& c_1\sin(0)e^{-3\cdot 0} + c_2\cos(0)e^{-3\cdot 0} \\ 6 &=& y'(0) &=& (-3c_1 - c_2)\sin(0)e^{-3\cdot 0} + (c_1 - 3c_2)\cos(0)e^{-3\cdot 0} \end{array} \tag{20}$$

$$\to \begin{array}{rcl} 0 &=& c_2 \\ 6 &=& c_1 - 3c_2 \end{array} \to \begin{array}{rcl} c_2 &=& & & 0 \\ c_1 &=& 6 + 3c_2 &=& 6 \end{array} \tag{21}$$

$$\to \boxed{y(t) = 6\sin(t)e^{-3t}}. \tag{22}$$

Problem 5.2.37: Solve the following initial value problem.

$$t^2y'' + 6ty' + 6y = 0; \quad y(1) = 0, y'(1) = -4. \tag{23}$$

Solution: We perform a substitution (or a change of variables) in order to convert equation (23) into a constant coefficient differential equation, which will then be straight-forward to solve. Letting $x = \ln(t)$, we see that $t = e^x$, and we may define $h(x) = y(e^x) = y(t)$. We see that

$$h'(x) = \frac{d}{dx}h(x) = \frac{d}{dx}y(e^x) = y'(e^x)\cdot\frac{d}{dx}e^x = y'(e^x)\cdot e^x = ty'(t), \text{ and} \tag{24}$$

$$h''(x) = \frac{d}{dx}h'(x) = \frac{d}{dx}(e^xy'(e^x)) = \frac{d}{dx}(e^x)\cdot y'(e^x) + e^x\cdot\frac{d}{dx}y'(e^x) \tag{25}$$

$$= e^xy'(e^x) + e^x\cdot e^xy''(e^x) = e^xy'(e^x) + e^{2x}y''(e^x) = ty'(t) + t^2y''(t). \tag{26}$$

We now see that

$$0 = t^2y'' + 6ty' + 6y = (t^2y'' + ty') + 5ty' + 6y \tag{27}$$

$$= (t^2y''(t) + ty'(t)) + 5ty'(t) + 6y(t) \tag{28}$$

$$= h''(x) + 5h'(x) + 6h(x) = h'' + 5h' + 6h. \tag{29}$$

We see that the characteristic equation of our converted equation is

$$0 = r^2 + 5r + 6 = (r+2)(r+3), \tag{30}$$

and has solutions $r = -3, -2$. It follows that the general solution to our converted equation is

$$h(x) = c_1e^{-2x} + c_2e^{-3x}. \tag{31}$$

Recalling that $x = \ln(t)$, we see that the general solution to equation (23) is

(32) $y(t) = h(x) = c_1e^{-2x} + c_2e^{-3x} = c_1e^{-2\ln(t)} + c_2e^{-3\ln(t)} = c_1t^{-2} + c_2t^{-3}$.

Making use of the initial conditions, we see that

(33)
$$\begin{array}{rcccccc} 0 &=& y(1) &=& c_1 \cdot 1^{-2} + c_2 \cdot 1^{-3} &=& c_1 + c_2 \\ -4 &=& y'(1) &=& -2c_1 \cdot 1^{-3} - 3c_2 \cdot 1^{-4} &=& -2c_1 - 3c_2 \end{array}$$

(34)
$$\rightarrow \begin{array}{rcrcr} c_1 &+& c_2 &=& 0 \\ -2c_1 &-& 3c_2 &=& -4 \end{array} \overset{R_2+2R_1}{\rightarrow} \begin{array}{rcrcr} c_1 &+& c_2 &=& 0 \\ && -c_2 &=& -4 \end{array}$$

(35)
$$\overset{\frac{1}{5}R_2}{\rightarrow} \begin{array}{rcrcr} c_1 &+& c_2 &=& 0 \\ && c_2 &=& 4 \end{array} \overset{R_1-R_2}{\rightarrow} \begin{array}{rcrcr} c_1 && &=& -4 \\ && c_2 &=& 4 \end{array}$$

(36)
$$\rightarrow \boxed{y(t) = -4t^{-2} + 4t^{-3}}.$$

Modified Problem 5.2.43: Determine A, ω, and φ for which

$$-3\sin(4t) + 3\cos(4t) = A\sin(\omega t + \varphi). \tag{37}$$

Solution: Firstly, we use the angle-addition formula for sin to see that

$$A\sin(\omega t + \varphi) = A\sin(\omega t)\cos(\varphi) + A\sin(\varphi)\cos(\omega t), \text{ so} \tag{38}$$

$$-3\sin(4t) + 3\cos(4t) = A\cos(\varphi)\sin(\omega t) + A\sin(\varphi)\cos(\omega t). \tag{39}$$

We now see that $\omega = 4$, and that

$$\begin{array}{rcl} A\cos(\varphi) & = & -3 \\ A\sin(\varphi) & = & 3 \end{array} \tag{40}$$

$$\to A^2 = A^2\cos^2(\varphi) + A^2\sin^2(\varphi) = (-3)^2 + 3^2 = 18 \to A = \pm 3\sqrt{2} \tag{41}$$

$$\to \begin{array}{rcl} \cos(\varphi) & = & \mp\frac{1}{\sqrt{2}} \\ \sin(\varphi) & = & \pm\frac{1}{\sqrt{2}} \end{array} \to \varphi = \frac{3\pi}{4}, -\frac{\pi}{4} \tag{42}$$

$$\to -3\sin(4t) + 3\cos(4t) = \underbrace{\boxed{3\sqrt{2}\sin(4t + \frac{3\pi}{4})}}_{\text{This is amplitude-phase form since A is positive.}} = \boxed{-3\sqrt{2}\sin(4t - \frac{\pi}{4})}. \tag{43}$$