Problem 1 (Not from the text book): Find the inverse of

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix}$$

Solution: We reduce the 3 by 6 matrix $[A|I_3]$ until the left half is in reduced echelon form, which will be I_3 since A is invertible.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \tag{1}$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \tag{2}$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right) \tag{3}$$

$$\xrightarrow{R_2 + \frac{5}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -5 & -2 & 5 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right) \xrightarrow{R_1 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 4 \\ 0 & 1 & 0 & -5 & -2 & 5 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right). \tag{4}$$

To check our work, we note that

$$AA^{-1} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & -1 & 4 \\ -5 & -2 & 5 \\ -2 & -1 & 2 \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} 1 \cdot (-3) + (-2) \cdot (-5) + 3 \cdot (-2) & 1 \cdot (-1) + (-2) \cdot (-2) + 3 \cdot (-1) & 1 \cdot 4 + (-2) \cdot 5 + 3 \cdot 2 \\ 0 \cdot (-3) + 2 \cdot (-5) + (-5) \cdot (-2) & 0 \cdot (-1) + 2 \cdot (-2) + (-5) \cdot (-1) & 0 \cdot 4 + 2 \cdot 5 + (-5) \cdot 2 \\ 1 \cdot (-3) + (-1) \cdot (-5) + 1 \cdot (-2) & 1 \cdot (-1) + (-1) \cdot (-2) + 1 \cdot (-1) & 1 \cdot 4 + (-1) \cdot 5 + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

Remark: We only have to check that $A^{-1}A = I_3$ **OR** $AA^{-1} = I_3$. We do not have to check both.

Problem 4.9.46: Consider the matrices A, D ane E given by

$$A^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \text{ and } E = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}. \tag{7}$$

Find matrices B and C for which $AB = D$ and $CA = E$.

Solution: We see that

$$A^{-1}D = A^{-1}(AB) = (A^{-1}A)B = I_2 B = B, \text{so} \tag{8}$$

$$B = A^{-1}D = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} 3\cdot(-1)+1\cdot 1 & 3\cdot 2+1\cdot 0 & 3\cdot 3+1\cdot 2 \\ 0\cdot(-1)+2\cdot 1 & 0\cdot 2+2\cdot 0 & 0\cdot 3+2\cdot 2 \end{bmatrix} \tag{10}$$

$$= \boxed{\begin{bmatrix} -2 & 6 & 11 \\ 2 & 0 & 4 \end{bmatrix}}. \tag{11}$$

Similarly, we see that

$$EA^{-1} = (CA)A^{-1} = C(AA^{-1}) = CI_2 = C, \text{ so} \tag{12}$$

$$C = EA^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 3+(-1)\cdot 0 & 2\cdot 1+(-1)\cdot 2 \\ 1\cdot 3+1\cdot 0 & 1\cdot 1+1\cdot 2 \\ 0\cdot 3+3\cdot 0 & 0\cdot 1+3\cdot 2 \end{bmatrix} \tag{13}$$

$$= \boxed{\begin{bmatrix} 6 & 0 \\ 3 & 3 \\ 0 & 6 \end{bmatrix}}. \tag{14}$$

Problem 4.9.59: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$, and let I_n denote the $(n \times n)$ identity matrix. Let $A = I_n + \vec{u}\vec{v}^T$, and suppose that $\vec{v}^T\vec{u} \neq -1$. Show that

$$A^{-1} = I_n - a\vec{u}\vec{v}^T, \text{ where } a = \frac{1}{1 + \vec{v}^T\vec{u}}. \tag{15}$$

This result is known as the Sherman-Woodberry formula.

Example: If $n = 3$,

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ then} \tag{16}$$

$$\vec{v}^T\vec{u} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (-1) \cdot 1 + 1 \cdot 2 + 0 \cdot 3 = 1 \neq -1 \text{ and} \tag{17}$$

$$A = I_3 + \vec{u}\vec{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \cdot (-1) & 1 \cdot 1 & 1 \cdot 0 \\ 2 \cdot (-1) & 2 \cdot 1 & 2 \cdot 0 \\ 3 \cdot (-1) & 3 \cdot 1 & 3 \cdot 0 \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{pmatrix}. \tag{20}$$

We also saw that

$$\vec{v}^T\vec{u} = 1 \text{ and } \vec{u}\vec{v}^T = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} \text{ so} \tag{21}$$

$$a = \frac{1}{1 + \vec{v}^T\vec{u}} = \frac{1}{1 + 1} = \frac{1}{2} \text{ and} \tag{22}$$

$$A^{-1} = I_3 - a\vec{u}\vec{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}. \tag{23}$$

Indeed, we see that

$$AA^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} 0\cdot\frac{3}{2}+1\cdot 1+0\cdot\frac{3}{2} & 0\cdot(-\frac{1}{2})+1\cdot 0+0\cdot(-\frac{3}{2}) & 0\cdot 0+1\cdot 0+0\cdot 1 \\ (-2)\cdot\frac{3}{2}+3\cdot 1+0\cdot\frac{3}{2} & (-2)\cdot(-\frac{1}{2})+3\cdot 0+0\cdot(-\frac{3}{2}) & (-2)\cdot 0+3\cdot 0+0\cdot 1 \\ (-3)\cdot\frac{3}{2}+3\cdot 1+1\cdot\frac{3}{2} & (-3)\cdot(-\frac{1}{2})+3\cdot 0+1\cdot(-\frac{3}{2}) & (-3)\cdot 0+3\cdot 0+1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

Solution: The inverse of a matrix (if it exists) is unique, so for

$$B = I_n - a\vec{u}\vec{v}^T, \tag{26}$$

we only have to verify that

$$AB = I_n \text{ or } BA = I_n, \tag{27}$$

as we will then know that A is invertible, and that $A^{-1} = B$. Since $\vec{v}^T\vec{u}$ is a scalar, let us simplify our notation by letting

$$b = \vec{v}^T\vec{u} \text{ so that } a = \frac{1}{1+b}. \tag{28}$$

We see that

$$AB = (I_n+\vec{u}\vec{v}^T)(I_n-a\vec{u}\vec{v}^T) = I_nI_n+\vec{u}\vec{v}^TI_n+I_n(-a\vec{u}\vec{v}^T)+\vec{u}\vec{v}^T(-a\vec{u}\vec{v}^T) \tag{29}$$

$$= I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a(\vec{u}\vec{v}^T)(\vec{u}\vec{v}^T) = I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a\vec{u}(\vec{v}^T\vec{u})\vec{v}^T \tag{30}$$

$$\overset{\text{By (28)}}{=} I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a\vec{u}(b)\vec{v}^T = I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - ab\vec{u}\vec{v}^T \tag{31}$$

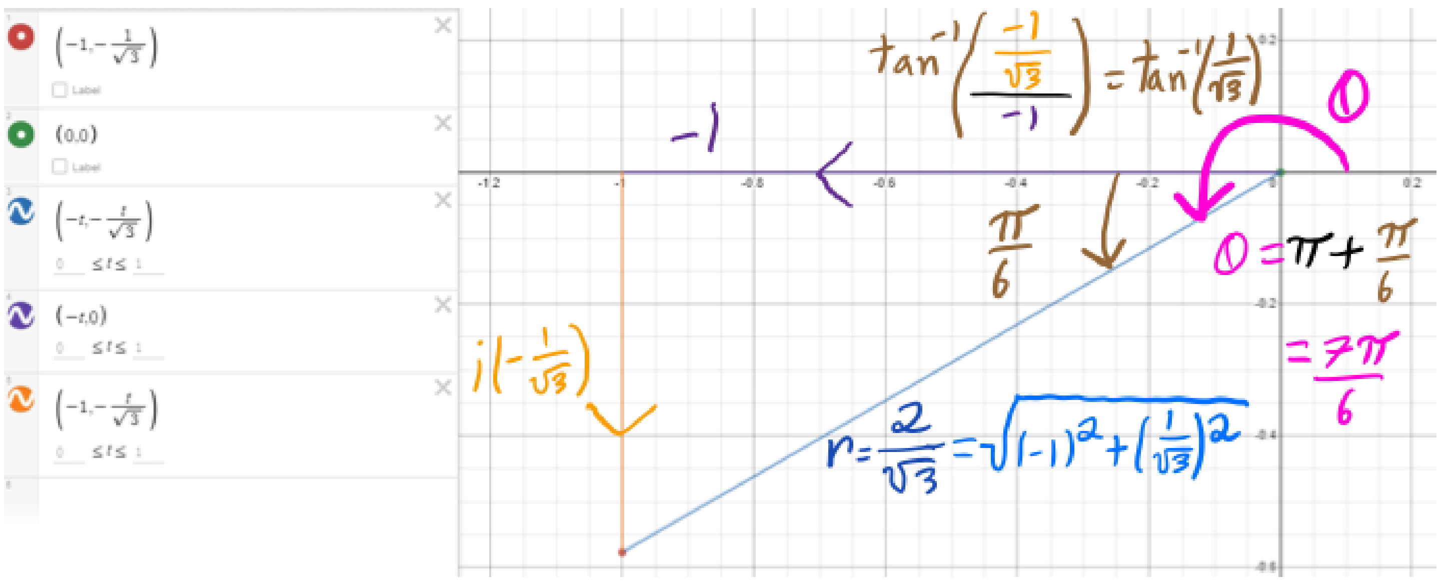
$$= I_n + (1 - a - ab)\vec{u}\vec{v}^T \overset{\text{By (28)}}{=} I_n + \left(1 - \frac{1}{1+b} - \frac{b}{1+b}\right)\vec{u}\vec{v}^T \tag{32}$$

$$= I_n + 0 \cdot \vec{u}\vec{v}^T = I_n. \tag{33}$$

Some Problems From the Appendix on Complex Numbers

Modified Problem 12: Plot $z = {\color{red}-}1 - \frac{1}{\sqrt{3}}i$ in the complex plane. Then find the modulus and argument of z, and express z in the form $z = re^{i\theta}$.

Solution: Based on the diagram below, we see that $-1 - \frac{1}{\sqrt{3}}i = \boxed{\frac{2}{\sqrt{3}}e^{i\frac{7\pi}{6}}}$.



Problem 19: For $z = -1 + 4i$ and $w = 5 + 2i$ evaluate $\left|\frac{z}{2w}\right|$.

Solution 1: We see that

$$\frac{z}{2w} = \frac{-1+4i}{2(5+2i)} = \frac{-1+4i}{10+4i} = \frac{-1+4i}{10+4i} \cdot \underbrace{\frac{10-4i}{10-4i}}_{1} = \frac{(-1+4i)(10-4i)}{(10+4i)(10-4i)} \tag{34}$$

$$= \frac{-10+40i+4i{\color{red}-16i^2}}{100+40i-40i{\color{red}-16i^2}} \overset{{\color{red}i^2=-1}}{=} \frac{-10+40i+4i{\color{red}+16}}{100+40i-40i{\color{red}+16}} = \frac{6+44i}{116} \tag{35}$$

$$= \frac{3+22i}{58} \to \left|\frac{z}{2w}\right| = \left|\frac{3+22i}{58}\right| = \frac{1}{58}|3+22i| = \frac{1}{58}\sqrt{3^2+22^2} = \boxed{\frac{\sqrt{493}}{58}}. \tag{36}$$

Solution 2: We see that

$$\left|\frac{z}{2w}\right| = \frac{|z|}{|2w|} = \frac{|z|}{2|w|} = \frac{|-1+4i|}{2|5+2i|} = \frac{\sqrt{(-1)^2+4^2}}{2\sqrt{5^2+2^2}} = \boxed{\frac{\sqrt{17}}{2\sqrt{29}} = \frac{\sqrt{493}}{58}}. \tag{37}$$

Problem 28: Evaluate $i(e^{i\frac{\pi}{6}} - e^{-i\frac{\pi}{6}})$.

Solution: Recalling Euler's formula

$$e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)), \text{ we see that} \tag{38}$$

$$i(e^{i\frac{\pi}{6}} - e^{-i\frac{\pi}{6}}) = i\left(\left(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})\right) - \left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)\right) \tag{39}$$

$$= i\left(\left(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})\right) - \left(\cos(\frac{\pi}{6}) - i\sin(\frac{\pi}{6})\right)\right) = i\left(2i\sin(\frac{\pi}{6})\right) \tag{40}$$

$$= i(2i \cdot \frac{1}{2}) = i^2 = \boxed{-1}. \tag{41}$$

Problem 53: Find all possible fourth roots of -16. Equivalently, find all possible values of $(-16)^{\frac{1}{4}}$.

Solution: We see that

$$-16 = 16 \cdot (-1) = 16e^{i\pi} = 16e^{i(\pi+2n\pi)} \text{ (where } n \text{ is an integer)} \tag{42}$$

$$\rightarrow (-16)^{\frac{1}{4}} = \left(16e^{i(\pi+2n\pi)}\right)^{\frac{1}{4}} = 16^{\frac{1}{4}}\left(e^{i(\pi+2n\pi)}\right)^{\frac{1}{4}} \tag{43}$$

$$= 2e^{i(\frac{\pi}{4}+\frac{n}{2}\pi)} \text{ (where } n \text{ is an integer)} \tag{44}$$

$$\rightarrow (-16)^{\frac{1}{4}} \in \{2e^{i\frac{\pi}{4}}, 2e^{i\frac{3\pi}{4}}, 2e^{i\frac{5\pi}{4}}, 2e^{i\frac{7\pi}{4}}\}. \tag{45}$$

Making use of Euler's formula, we see that

$$2e^{i\frac{\pi}{4}} = 2\left(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right) = 2(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \sqrt{2} + \sqrt{2}i, \tag{46}$$

$$2e^{i\frac{3\pi}{4}} = 2\left(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\right) = 2(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = -\sqrt{2} + \sqrt{2}i, \tag{47}$$

$$2e^{i\frac{5\pi}{4}} = 2\left(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})\right) = 2(-\frac{1}{\sqrt{2}} + i(-\frac{1}{\sqrt{2}})) = -\sqrt{2} - \sqrt{2}i, \tag{48}$$

$$2e^{i\frac{7\pi}{4}} = 2\left(\cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4})\right) = 2(\frac{1}{\sqrt{2}} + i(-\frac{1}{\sqrt{2}})) = \sqrt{2} - \sqrt{2}i, \tag{49}$$

$$\to (-16)^{\frac{1}{4}} \in \boxed{\{\sqrt{2} + \sqrt{2}i, -\sqrt{2} + \sqrt{2}i, -\sqrt{2} - \sqrt{2}i, \sqrt{2} - \sqrt{2}i\}}. \tag{50}$$