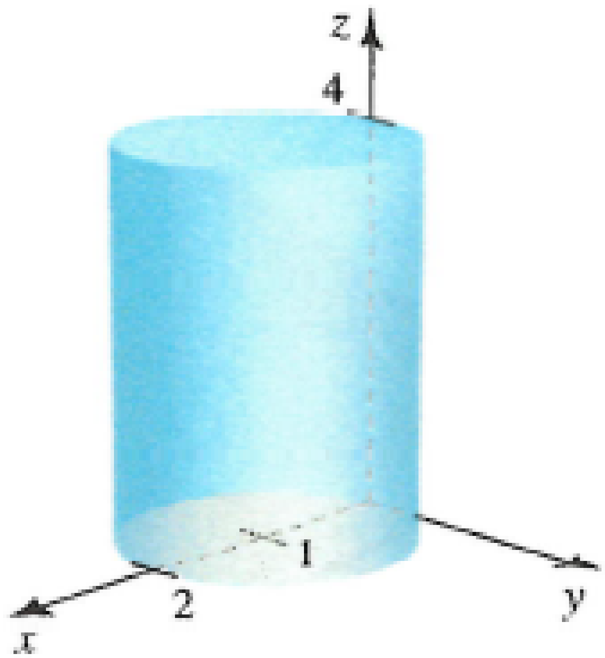
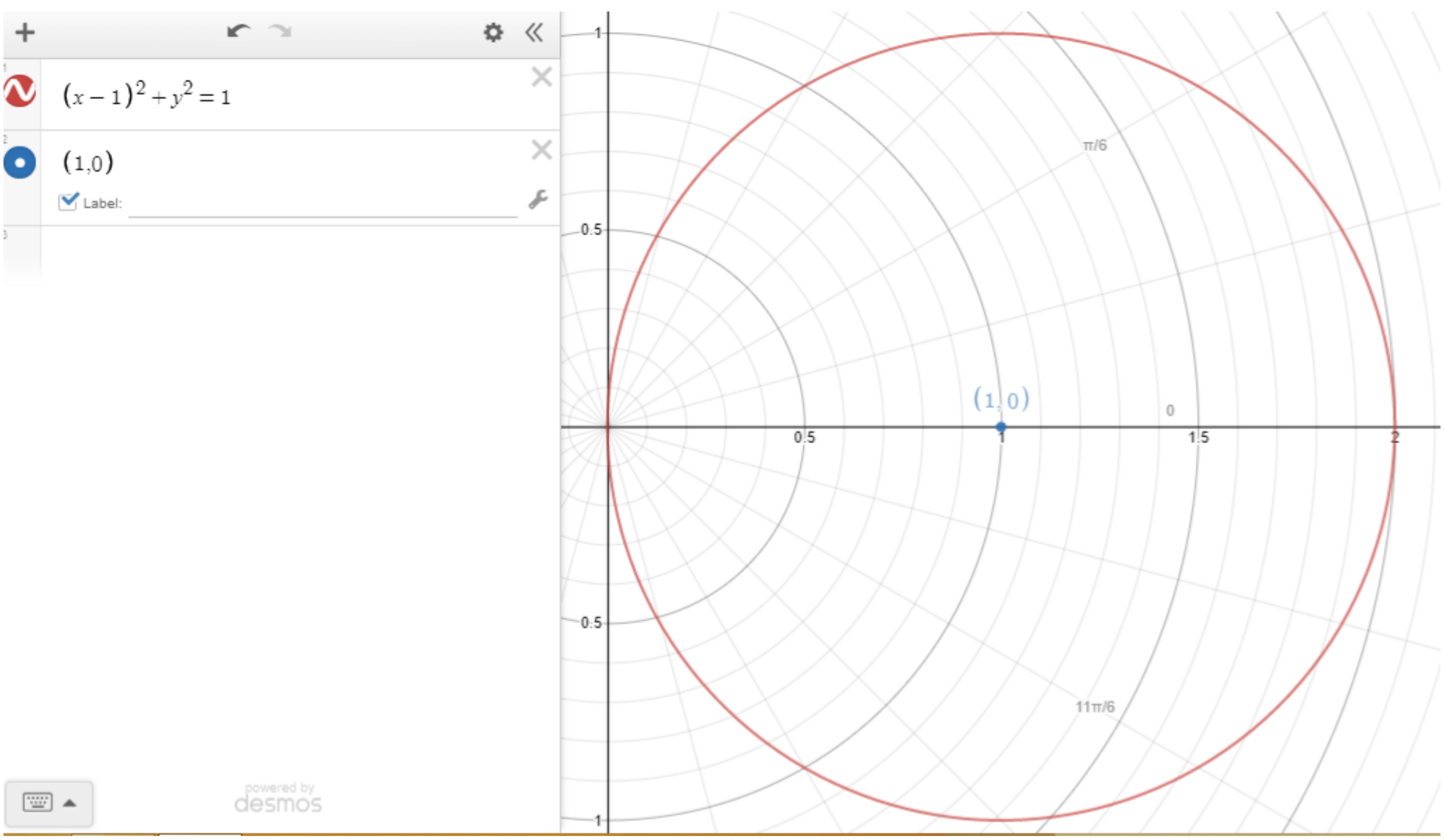
Problem 2.45: Find the volume of the solid cylinder E whose height is 4 and whose base is the disk $\{(r, \theta) : 0 \leq r \leq 2\cos(\theta)\}$.



Solution: We first look at the cross section of E in the xy−plane to help us determine our bounds.



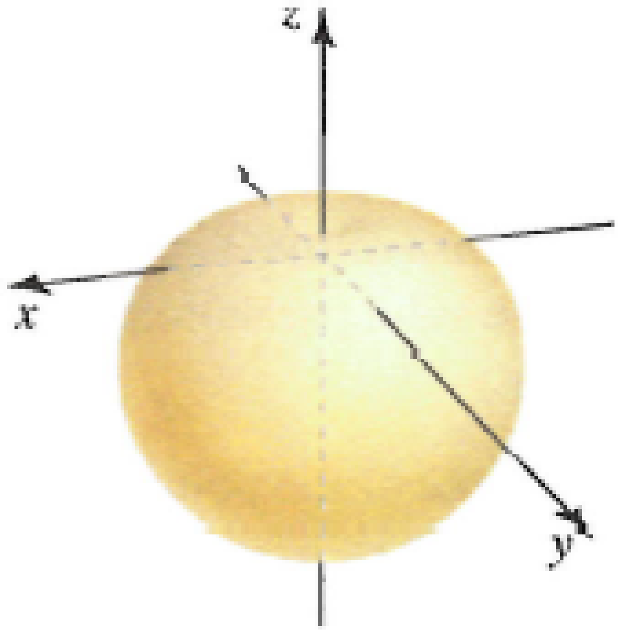
$$\text{Volume}(E) = \iiint_E 1dV = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} rdrd\theta dz \tag{1}$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_0^{2\cos(\theta)} d\theta dz = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2(\theta)d\theta dz \tag{2}$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(2\theta)+1)d\theta dz = \int_0^4 (\frac{1}{2}\sin(2\theta)+\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \tag{3}$$

$$= \int_0^4 \pi dz = \boxed{4\pi}. \tag{4}$$

Problem 2.48: Find the volume of the solid cardiod of revolution $D = \{(\rho, \varphi, \theta) : 0 \le \rho \le \frac{1}{2}(1 - \cos(\varphi)), 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$.



Solution: In this problem, the description of the region is just a reordering of the description that we need to write down our triple integral in spherical coordinates to find the volume. We see that

$$\text{Volume}(D) = \iiint_D 1dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{1}{2}(1-\cos(\varphi))} \rho^2 \sin(\varphi) d\rho d\varphi d\theta \tag{5}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3 \sin(\varphi)\Big|_0^{\frac{1}{2}(1-\cos(\varphi))} d\varphi d\theta \tag{6}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\left(\underbrace{\frac{1}{2}(1-\cos(\varphi))}_{u}\right)^3 \underbrace{\sin(\varphi)d\varphi}_{2du} d\theta = \int_0^{2\pi} \frac{1}{6}u^4\Big|_{\varphi=0}^{\pi} d\theta \tag{7}$$

$$= \int_0^{2\pi} \frac{1}{6}\left(\frac{1}{2}(1-\cos(\varphi))\right)^4\Big|_0^{\pi} d\theta = \int_0^{2\pi} \frac{1}{6} d\theta = \boxed{\frac{\pi}{3}}. \tag{8}$$

Problem 3.26: Consider the vector field $\vec{F} = \langle x, -y\rangle$ and the curve C which is the square with vertices $(\pm 1, \pm 1)$ with the counterclockwise orientation.

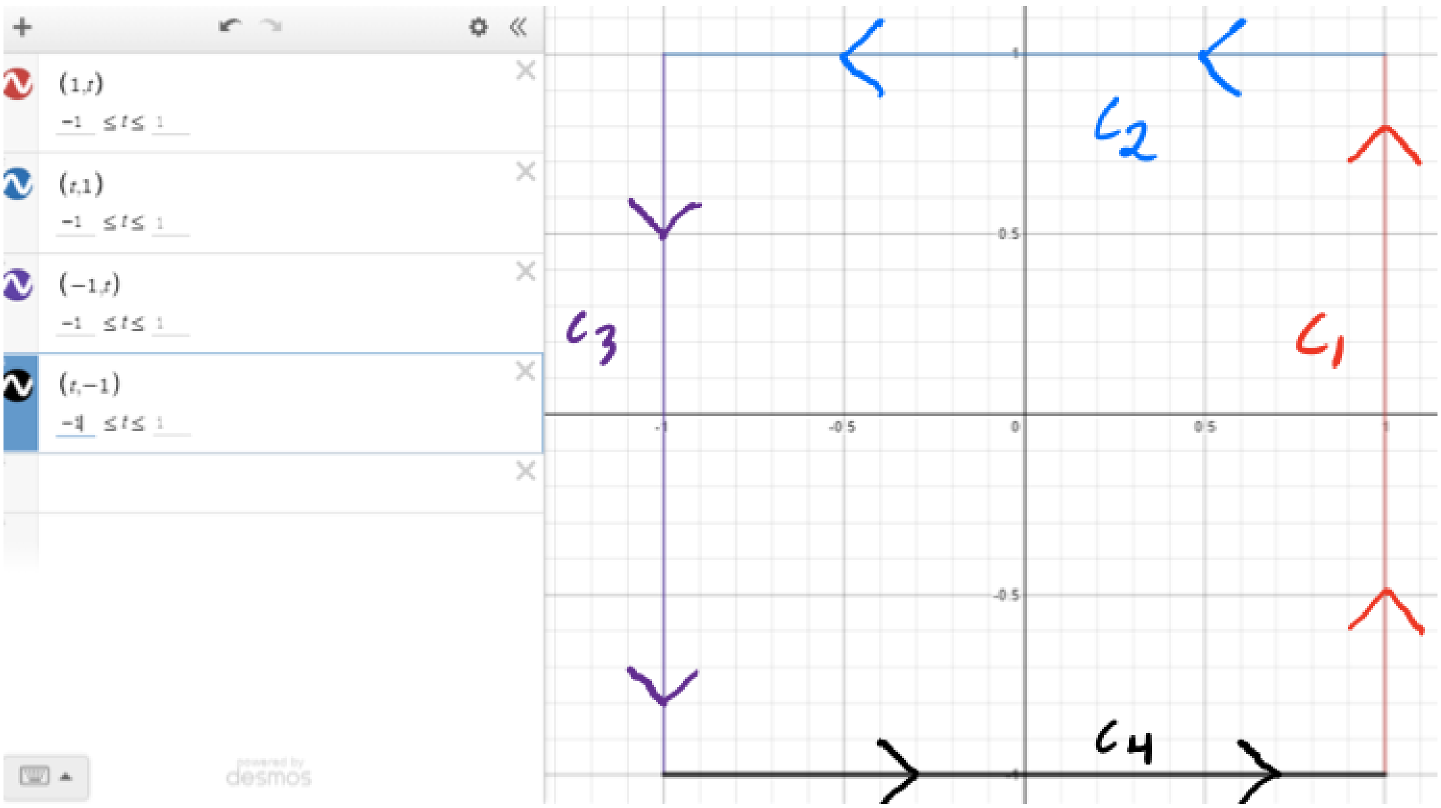


FIGURE 1. The curve C.

a) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by finding a parameterization $\vec{r}(t)$ for the curve C.
b) By using the Fundamental Theorem for Line Integrals.

Solution to a: Letting C_1, C_2, C_3, and C_4 be as in Figure 1, we see that

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}. \tag{9}$$

Since

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-1}^{1}\langle 1, -t\rangle\cdot\langle 0, 1\rangle dt = \int_{-1}^{1} -t dt = -\frac{1}{2}t^2\Big|_{-1}^{1} = 0, \tag{10}$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{1}^{-1}\langle t, -1\rangle\cdot\langle 1, 0\rangle dt = \int_{1}^{-1} t dt = \frac{1}{2}t^2\Big|_{1}^{-1} = 0, \tag{11}$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_1^{-1} \langle -1, -t \rangle \cdot \langle 0, 1 \rangle dt = \int_1^{-1} -tdt = -\frac{1}{2}t^2\Big|_1^{-1} = 0, \tag{12}$$

$$\int_{C_4} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \langle t, 1 \rangle \cdot \langle 1, 0 \rangle dt = \int_{-1}^{1} tdt = \frac{1}{2}t^2\Big|_{-1}^{1} = 0, \tag{13}$$

we see that

$$\int_C \vec{F} \cdot d\vec{r} = 0 + 0 + 0 + 0 = \boxed{0}. \tag{14}$$

Solution to b: Since

$$\frac{\partial}{\partial y}(x) = 0 = \frac{\partial}{\partial x}(-y), \tag{15}$$

we see that $\vec{F} = \langle x, -y \rangle$ is a conservative vector field. We now have 2 ways in which to finish the problem.

Finish 1: Since $\vec{F}$ is a conservative vector field and C is a (simple, piecewise smooth, oriented) closed curve, we see that

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}. \tag{16}$$

Finish 2: We now want to find a potential function $\varphi(x, y)$ for $\vec{F}$. Since

$$\langle \varphi_x, \varphi_y \rangle = \nabla \varphi = \vec{F} = \langle x, -y \rangle, \tag{17}$$

we see that

$$\varphi_x(x, y) = x \to \varphi(x, y) = \int xdx = \frac{1}{2}x^2 + g(y) \to \tag{18}$$

$$g'(y) = \varphi_y(x, y) = -y \to g(y) = -\frac{1}{2}y^2 + C \to \varphi(x, y) = \frac{1}{2}(x^2 - y^2) + C. \tag{19}$$

Now let P be any point on the curve C. For example, we may take $P = (1, 1)$. Since the curve C can be seen as starting at P and ending at P, the Fundamental Theorem for Line Integrals tells us that

$$\int_C \vec{F} \cdot d\vec{r} = \varphi\left((1,1)\right) - \varphi\left((1,1)\right) = \boxed{0}. \tag{20}$$

Remark: We see that in Finish 2, we did not even need to determine what the function φ was in order to conclude that the final answer is 0.

Problem 4.2: Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -3 & 1 & -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{21}$$

a) Determine conditions on b_1, b_2, and b_3 that are necessary and sufficient for the system of equations $A\vec{x} = \vec{b}$ to be consistent.

b) For each of the following choices of $\vec{b}$, either show that the system $A\vec{x} = \vec{b}$ is inconsistent or exhibit the solution.

$$\textbf{i) } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ \textbf{ii) } \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \ \textbf{iii) } \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \ \textbf{iv) } \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution to a: We begin by representing the equation $A\vec{x} = \vec{b}$ as an augmented matrix that we will proceed to row reduce into reduced echelon form.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & b_1 \\ 2 & -1 & 1 & b_2 \\ -3 & 1 & -3 & b_3 \end{array}\right] \overset{\substack{R_2 - 2R_1 \\ R_3 + 3R_1}}{\to} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & b_1 & & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & -2 & -6 & 3b_1 & & +b_3 \end{array}\right] \tag{22}$$

$$\overset{R_3+2R_2}{\to} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & b_1 & & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & 0 & 0 & -b_1 & +2b_2 & +b_3 \end{array}\right] \tag{23}$$

At this point you can already deduce when the system is consistent.

$$\overset{R_1+R_2}{\to} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -b_1 & +b_2 & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & 0 & 0 & -b_1 & +2b_2 & +b_3 \end{array}\right] \tag{24}$$

From the third row of the augmented matrix in equation (24), we see that

$$-b_1 + 2b_2 + b_3 = 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0, \tag{25}$$

and that the system of equations $A\vec{x} = \vec{b}$ is consistent if and only if equation (25) is true. Furthermore, in the event that equation (25) is true, we see that equations represented in equation (24) are

$$\begin{array}{ccccccccc} x_1 & & & + & 2x_3 & = & -b_1 & + & b_2 \\ & & x_2 & + & x_3 & = & -2b_1 & + & b_2 \end{array} \tag{26}$$

$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free.} \tag{27}$$

Solution to b: In part **a** we obtained a formula for $\vec{x}$ in terms of $\vec{b}$, so we will now apply that formula to each of the vectors.

$$\textbf{i: } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 2 \neq 0 \rightarrow \boxed{\text{The system is inconsistent}}.$$

$$\textbf{ii: } \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 0$$

$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free} \tag{28}$$

$$\rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3 \\ -x_3 - 8 \\ x_3 \end{bmatrix}, x_3 \text{ is free}}. \tag{29}$$

$$\textbf{iii: } \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 0$$

$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free} \tag{30}$$

$$\rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 - 4 \\ -x_3 - 11 \\ x_3 \end{bmatrix}, x_3 \text{ is free}}. \tag{31}$$

iv: $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 4 \neq 0 \rightarrow \boxed{\text{The system is inconsistent}}$.