Problem 3.2.31: Use a scalar line integral to find the length of the curve

$$\vec{r}(t) = \langle 20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2}\rangle, \text{ for } 0 \le t \le 2. \tag{1}$$

Solution: Firstly, we note that

$$\vec{r}\,'(t) = \langle 5\cos(\frac{t}{4}), -5\sin(\frac{t}{4}), \frac{1}{2}\rangle. \tag{2}$$

We now see that

$$\text{Arclength(C)} = \int_C 1ds = \int_0^2 |\vec{r}\,'(t)|dt = \int_0^2 |\langle 5\cos(\frac{t}{4}), -5\sin(\frac{t}{4}), \frac{1}{2}\rangle|dt \tag{3}$$

$$= \int_0^2 \sqrt{\left(5\cos(\frac{t}{4})\right)^2 + \left(-5\sin(\frac{t}{4})\right)^2 + \left(\frac{1}{2}\right)^2} dt \tag{4}$$

$$= \int_0^2 \sqrt{25\cos^2(\frac{t}{4}) + 25\sin^2(\frac{t}{4}) + \frac{1}{4}} dt = \int_0^2 \sqrt{25\frac{1}{4}} dt \tag{5}$$

$$= \sqrt{25\frac{1}{4}}t\Big|_0^2 = 2\sqrt{25\frac{1}{4}} = \boxed{\sqrt{101}}. \tag{6}$$

Problem 3.2.46: Find the work required to move an object along the line segment from $(1,1,1)$ to $(8,4,2)$ through the forcefield $\vec{F}$ given by

$$\vec{F} = \frac{\langle x, y, z\rangle}{x^2+y^2+z^2}. \tag{7}$$

Solution 1: Firstly, we recall that one method of parameterizing the line segment that starts at $\vec{p}$ and ends at $\vec{q}$ is to use the parameterization

$$\vec{r}(t) = (1-t)\vec{p} + t\vec{q} = \vec{p} + t(\vec{q}-\vec{p}), \quad 0 \le t \le 1. \tag{8}$$

It follows that

$$\vec{r}(t) = \langle 1,1,1\rangle + t\left(\langle 8,4,2\rangle - \langle 1,1,1\rangle\right) = \langle 1+7t, 1+3t, 1+t\rangle, \quad 0 \le t \le 1, \tag{9}$$

is a parameterization of the line segment from $(1,1,1)$ to $(8,4,2)$. We now see that

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)dt \tag{10}$$

$$= \int_0^1 \underbrace{\frac{\langle 1+7t, 1+3t, 1+t\rangle}{(1+7t)^2+(1+3t)^2+(1+t)^2}}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle 7,3,1\rangle dt}_{d\vec{r}} \tag{11}$$

$$= \int_0^1 \frac{(1+7t)\cdot 7 + (1+3t)\cdot 3 + (1+t)\cdot 1}{1+14t+49t^2+1+6t+9t^2+1+2t+t^2}dt \tag{12}$$

$$= \int_0^1 \frac{11+59t}{3+22t+59t^2}dt = \int_0^1 \frac{t+\frac{11}{59}}{t^2+\frac{22}{59}t+\frac{3}{59}}dt = \int_0^1 \frac{t+\frac{11}{59}}{(t+\frac{11}{59})^2+\frac{56}{3481}}dt \tag{13}$$

$$= \frac{1}{2}\ln\left((t+\frac{11}{59})^2+\frac{56}{3481}\right)\Big|_0^1 = \boxed{\frac{1}{2}\ln(28)}. \tag{14}$$

Solution 2: We note that for $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$ we have $\nabla\varphi = \vec{F}$, so the Fundamental Theorem for Line Integrals (section 3.3) allows us to simplify the calculations from equations (10)-(14) as follows.

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi\cdot d\vec{r} = \varphi\left((8,4,2)\right) - \varphi\left((1,1,1)\right) \tag{15}$$

$$= \frac{1}{2}\ln(8^2+4^2+2^2) - \frac{1}{2}\ln(1^2+1^2+1^2) = \frac{1}{2}\ln(84) - \frac{1}{2}\ln(3) = \boxed{\frac{1}{2}\ln(28)}. \tag{16}$$

Problem (not from the book): Determine whether the vector field $\vec{F}$ given by

$$\vec{F} = \langle y - e^{x+y}, x - e^{x+y} + 1, \frac{1}{z} \rangle \tag{17}$$

is a conservative vector field. If $\vec{F}$ is conservative, then find a potential function φ.

Solution: Letting

$$m(x,y,z) = y - e^{x+y}, \quad n(x,y,z) = x - e^{x+y} + 1, \quad p(x,y,z) = \frac{1}{z}, \tag{18}$$

we see that

$$\vec{F} = \langle m, n, p \rangle, \quad \text{and} \tag{19}$$

$$\frac{\partial m}{\partial y} = 1 - e^{x+y} = \frac{\partial n}{\partial x}, \quad \frac{\partial n}{\partial z} = 0 = \frac{\partial p}{\partial y}, \quad \frac{\partial m}{\partial z} = 0 = \frac{\partial p}{\partial x}, \tag{20}$$

so $\vec{F}$ is a conservative vector field, so we will now find the potential function φ. We recall that

$$\langle m, n, p \rangle = \vec{F} = \nabla \varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle. \tag{21}$$

We will now handle the 3 scalar differential equations that arise from (21) in order to find φ (but only up to a constant).

$$\varphi_x(x,y,z) = m(x,y,z) = y - e^{x+y} \to \varphi(x,y,z) = xy - e^{x+y} + h(y,z). \tag{22}$$

$$x - e^{x+y} + 1 = n(x,y,z) = \varphi_y(x,y,z) = x - e^{x+y} + h_y(y,z) \tag{23}$$
$$\to h_y(y,z) = 1 \to h(y,z) = y + g(z) \to \varphi(x,y,z) = xy - e^{x+y} + y + g(z).$$

$$\frac{1}{z} = p(x,y,z) = \varphi_z(x,y,z) = g_z(z) = g'(z) \to g(z) = \ln|z| + C$$
$$\to \boxed{\varphi(x,y,z) = xy - e^{x+y} + y + \ln|z| + C}. \tag{24}$$

Problem (not from the book): Evaluate

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r}, \tag{25}$$

where C is the curve that is shown in the picture below.

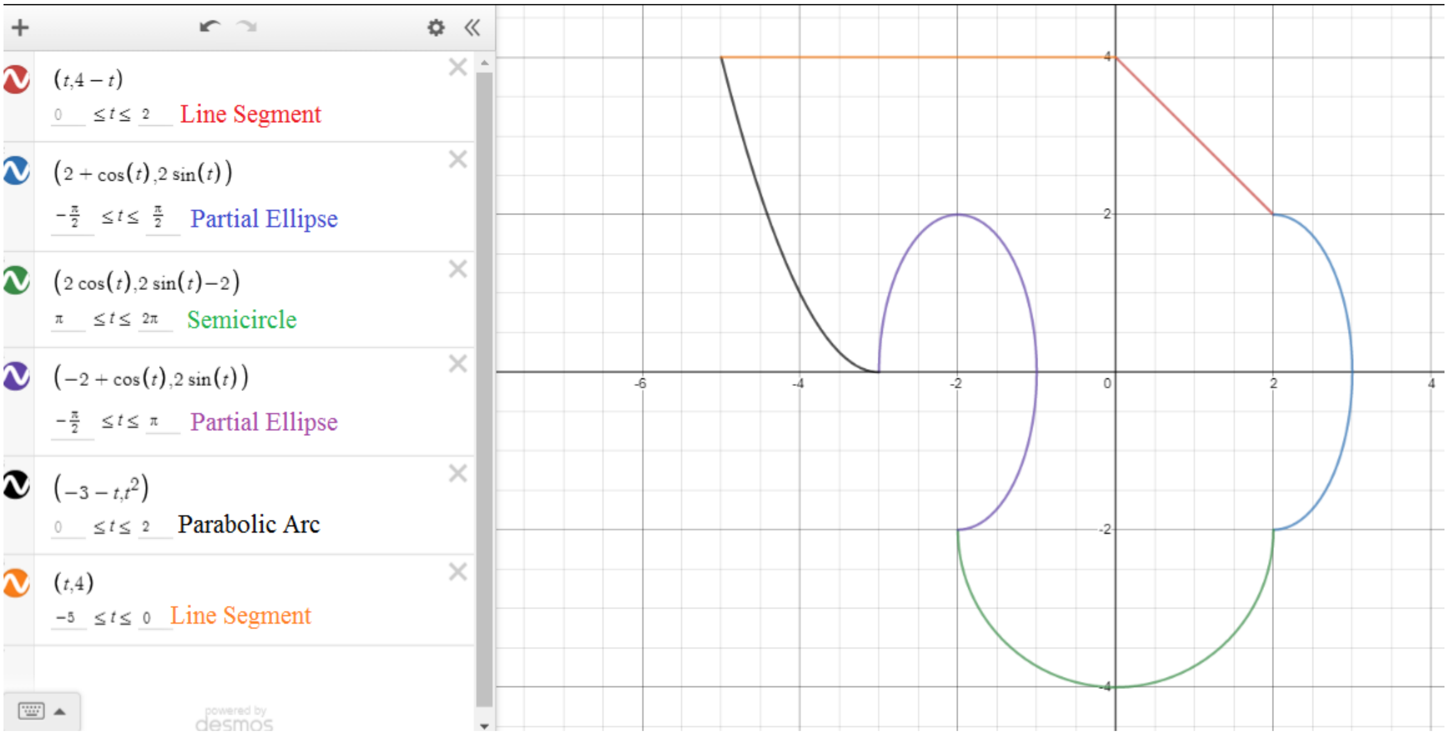

FIGURE 1

Solution: Letting

$$m(x,y,z) = \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, \text{ and} \tag{26}$$

$$n(x,y,z) = y^3 + 2 + e^{y^2}, \text{ we see that} \tag{27}$$

$$\vec{F} := \langle m, n \rangle, \text{ satisfies} \tag{28}$$

$$\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x} \tag{29}$$

so $\vec{F}$ is a conservative vector field. We also see that

$$(30) \quad \int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}.$$

Since $\vec{F}$ is conservative and C is a (simple piecewise smooth oriented) closed curve, we see that

$$(31) \quad \int_C \vec{F} \cdot d\vec{r} = \boxed{0}.$$

Challenge for the brave: Letting C once again denote the curve in figure 1, evaluate

$$(32) \quad \int_C \langle y, 0 \rangle \cdot d\vec{r}.$$