Problem 2.5.49: Find the volume of the solid region S outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos(\varphi)$.

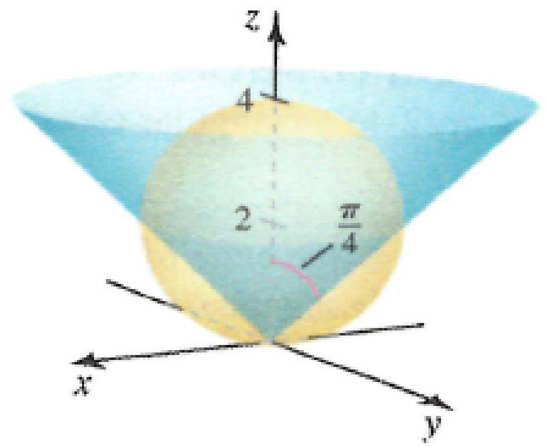

FIGURE 1. From page 167 of the textbook.

First Solution: We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. Remember that the angle φ is measured from the z-axis and satisfies $0 \leq \varphi \leq \pi$, not $0 \leq \varphi \leq 2\pi$. Also remember that this cross section is showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

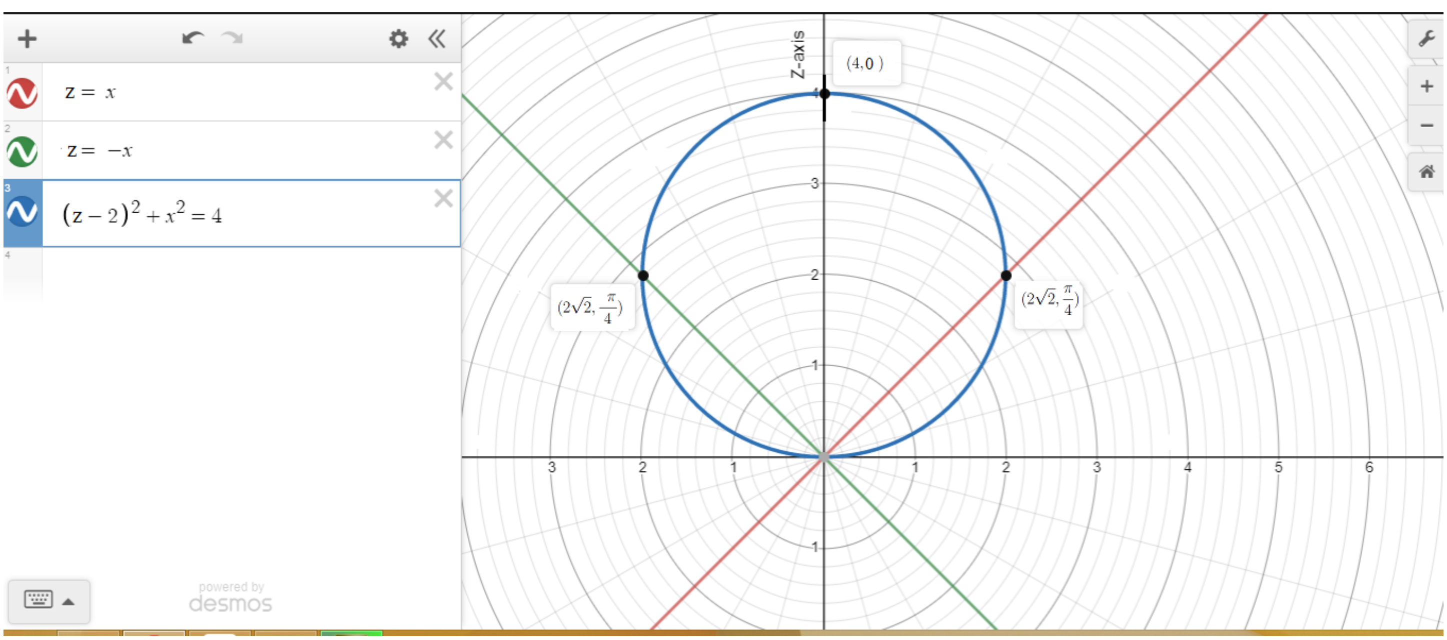

FIGURE 2. The xz-plane cross section in spherical coordinates.

We now see that for any $\theta \in [0, 2\pi)$ we have that $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$. Recalling that the blue circle is defined by $\rho = 4\cos(\varphi)$, we see that once φ is also chosen we have that $0 \leq \rho \leq 4\cos(\varphi)$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1 dV = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4\cos(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{1}$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=0}^{4\cos(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{64}{3}\underbrace{\cos^3(\varphi)}_{u^3}\underbrace{\sin(\varphi)d\varphi}_{-du}d\theta \tag{2}$$

$$= -\frac{64}{3}\int_0^{2\pi}\int_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} u^3 du d\theta = -\frac{64}{3}\int_0^{2\pi}\frac{1}{4}u^4\Big|_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \tag{3}$$

$$= -\frac{64}{3}\int_0^{2\pi}\frac{1}{4}\cos^4(\varphi)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = -\frac{64}{3}\int_0^{2\pi} -\frac{1}{16}d\theta = -\frac{64}{3}\cdot 2\pi \cdot \frac{-1}{16} = \boxed{\frac{8\pi}{3}}. \tag{4}$$

Second Solution: We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). Remember that this cross section is also showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

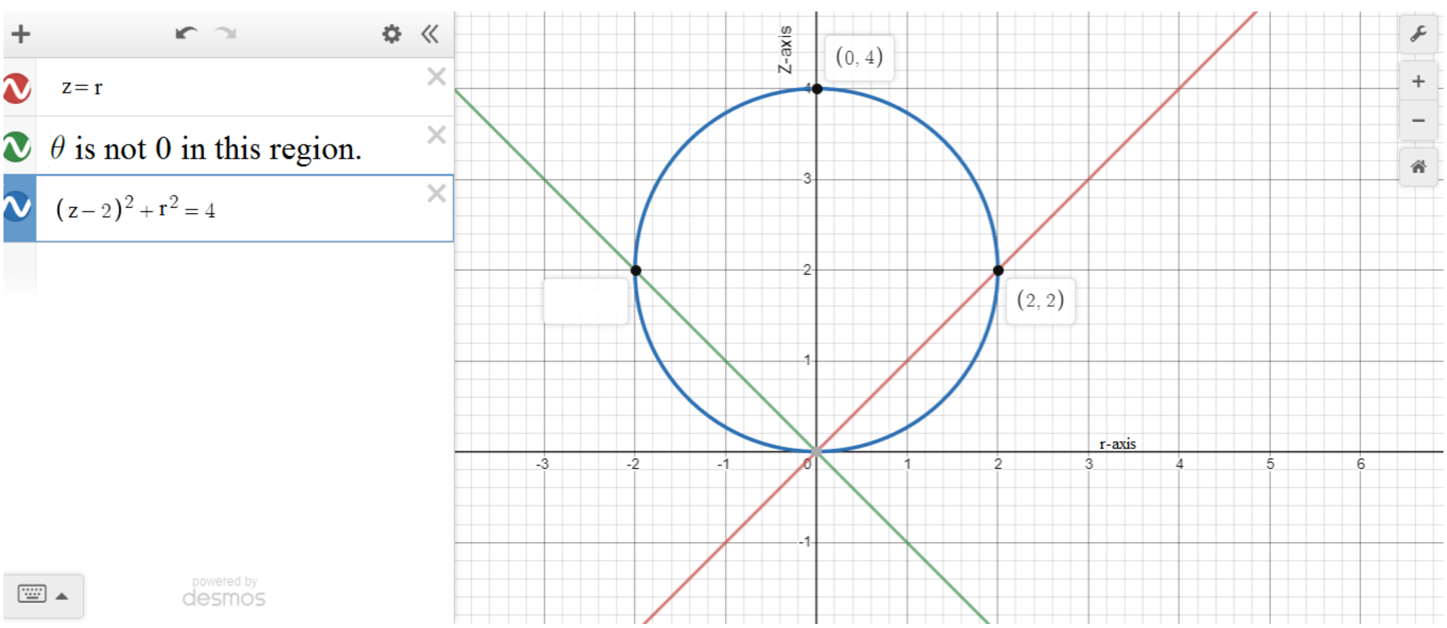

FIGURE 3. The xz-plane cross section in cylindrical coordinates.

We now see that for any $0 \leq \theta < 2\pi$ we have that $0 \leq z \leq 2$. Noting that we have $r = \sqrt{4-(z-2)^2} = \sqrt{4z-z^2}$ on the blue circle, we see that once z is chosen we have $z \leq r \leq \sqrt{4z-z^2}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_0^2\int_z^{\sqrt{4z-z^2}} rdrdzd\theta \tag{5}$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^2\Big|_z^{\sqrt{4z-z^2}} dzd\theta = \int_0^{2\pi}\int_0^2 (2z-z^2)dzd\theta \tag{6}$$

$$\int_0^{2\pi} (z^2 - \frac{1}{3}z^3)\Big|_0^2 d\theta = \int_0^{2\pi} \frac{4}{3}d\theta = \boxed{\frac{8\pi}{3}}. \tag{7}$$

Problem 2.5.50: Find the volume of the solid region S that is bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$.

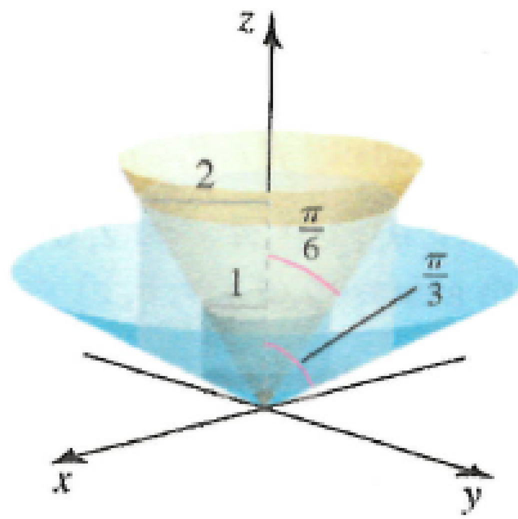

FIGURE 4. From page 167 of the textbook.

First Solution: We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

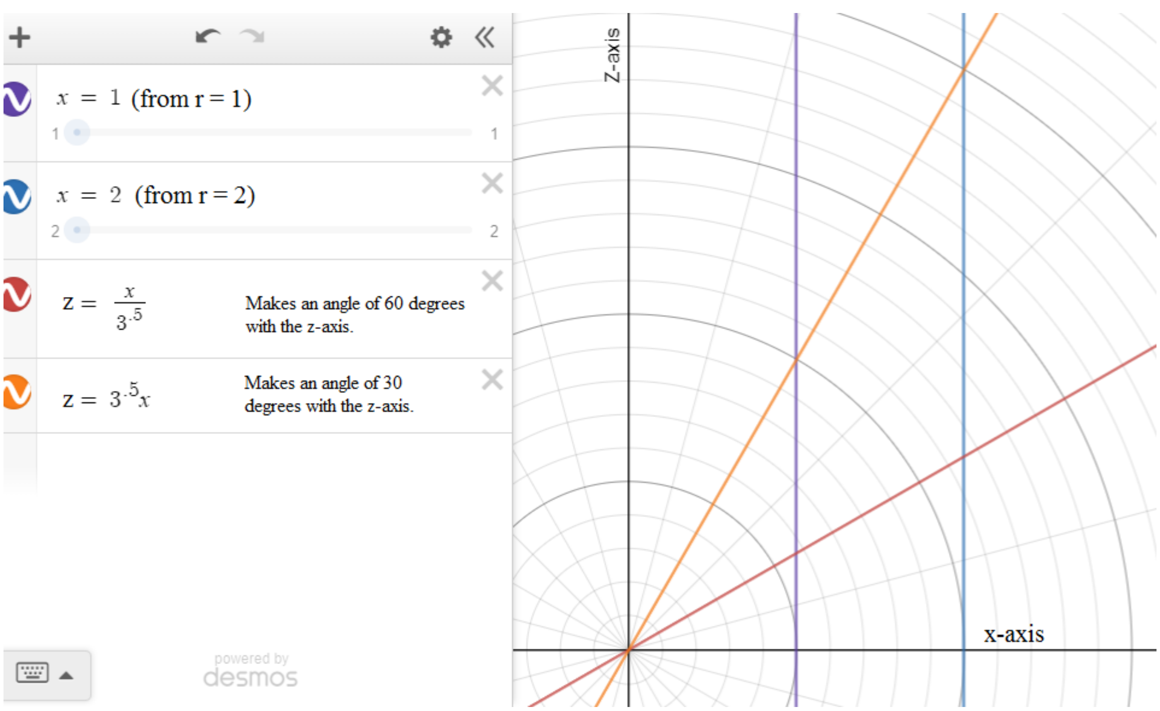

FIGURE 5. The xz-plane cross section in spherical coordinates.

We see that for any $0 \leq \theta < 2\pi$ we have $\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3}$. Noting that $r = \rho\sin(\varphi)$, we see that when $r = 1$ we have $\rho = \csc(\varphi)$ and when $r = 2$ we have $\rho = 2\csc(\varphi)$. It follows that once φ is also chosen we have $\csc(\varphi) \leq \rho \leq 2\csc(\varphi)$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\csc(\varphi)}^{2\csc(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{8}$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=\csc(\varphi)}^{2\csc(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{7}{3}\csc^2(\varphi)d\varphi d\theta \tag{9}$$

$$= \int_0^{2\pi} -\frac{7}{3}\cot(\varphi)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta = \int_0^{2\pi} \frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{10}$$

Second Solution: We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

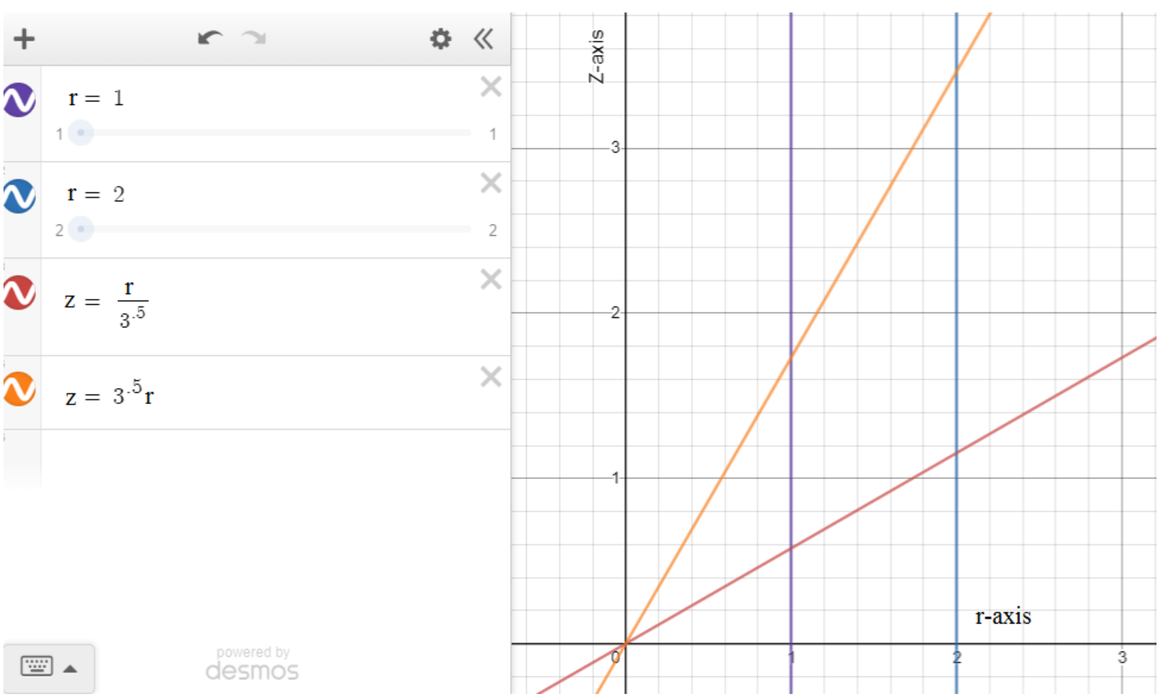

FIGURE 6. The xz-plane cross section in cylindrical coordinates.

We note that for any $0 \leq \theta < 2\pi$ we have $1 \leq r \leq 2$. Once r is also chosen, we see that $\frac{1}{\sqrt{3}}r \leq z \leq r\sqrt{3}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_1^2\int_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}} rdzdrd\theta \tag{11}$$

$$= \int_0^{2\pi}\int_1^2 rz\Big|_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}}drd\theta = \int_0^{2\pi}\int_1^2 \frac{2}{\sqrt{3}}r^2drd\theta = \int_0^{2\pi}\frac{2}{3\sqrt{3}}r^3\Big|_1^2 d\theta \tag{12}$$

$$= \int_0^{2\pi}\frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{13}$$