1. Problem 4.6.42 (Parts a and b)

Part a: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

In particular, for 2 by 2 matrices, the transpose merely switches the positions of b and c. It follows that $(A^T)^T = A$. We also see that

$$C = A^T + B \Rightarrow A^T = C - B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

It folows that

$$A = (A^T)^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}. \square$$

Part b: Once again let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

and note that

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = A^T B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} a+c & 3a+4c \\ b+d & 3b+4d \end{bmatrix}. \tag{1.1}$$

Focusing on the upper 2 entries of the left and right hand sides of equation (1.1), we obtain the system of equations

$$a + c = 2$$
$$3a + 4c = 3.$$

From Eq2-3*Eq1, we obtain the system

$$a + c = 2$$

$$c = -3 \Rightarrow a = 2 - c = 5,$$

So we know that $a = 5$ and $c = -3$ which gives us half of the entries of A. To solve or b and d we merely focus our attention on the lower 2 entries of equation (1.1) to obtain the system of equations

$$\begin{aligned} b + d &= 4 \\ 3b + 4d &= 5. \end{aligned}$$

Once again, from Eq2-3*Eq1, we obtain the system

$$\begin{aligned} b + d &= 4 \\ d = -7 \Rightarrow b &= 4 - d = 11, \end{aligned}$$

so $b = 11$ and $d = -7$. In conclusion,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ -3 & -7 \end{bmatrix}. \square$$

2. Problem 4.6.28

We know that

$$AB = A^2 \Rightarrow 0 = A^2 - AB = A(A - B).$$

The problem is in the next step. Even though A is not the 0 matrix, we cannot 'divide' by our nonzero matrix A the same way that we can divide by a nonzero scalar, so we cannot immediately assert that $A - B = 0$. In fact, unlike real numbers, there exists pairs of nonzero matricies that multiply to 0. Consider for example

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot (-1) & 1 \cdot 1 + 1 \cdot (-1) \\ 1 \cdot 1 + 1 \cdot (-1) & 1 \cdot 1 + 1 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} -9 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-9) + 3 \cdot 3 & 1 \cdot 3 + 3 \cdot (-1) \\ 2 \cdot (-9) + 6 \cdot 3 & 2 \cdot 3 + 6 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. Problems 4.7.16-4.7.27

Problem 4.7.16:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \overset{R2-3R1}{\Longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \overset{-\frac{1}{2}R2}{\Longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \overset{R1-2R2}{\Longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since A is row equivalent to I_2 (the 2 by 2 identity matrix) we see that A is nonsingular. □

Problem 4.7.17:

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \overset{R2-2R1}{\Longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Since the reduced echelon form of B contains a row of 0s, B is not row equivalent to I_2, so B is a singular matrix. Now let us find the null space of B. If

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is such that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = B\vec{x},$$

Then we also have that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 0 \end{bmatrix}.$$

It follows that $x_1 + 2x_2 = 0$, so $x_1 = -2x_2$. It follows that the null space of B is

$$\left\{ \begin{bmatrix} -2x \\ x \end{bmatrix} \mid x \in \mathbb{R} \right\}. \square$$

Problem 4.7.18: $C = A^T$, so C is also nonsingular. The transpose of a matrix never changes whether it is or is not singular.

Problem 4.7.19: AB is singular because B is singular. In fact, AB and B even have the same null space! To see this, we note that if $\vec{v}$ is in the null space of B, then

$$(AB)\vec{v}) = A(B\vec{v}) = A(\vec{0}) = \vec{0},$$

so $\vec{v}$ is in the null space of AB. Furthermore, if $\vec{v}$ is not in the null space of B, then $B\vec{v}$ is not the 0 vector, so $A(B\vec{v})$ is also not the 0 vector because A is nonsingular, so $\vec{v}$ is not in the null space of AB. Recalling problem 4.7.17, we see that the null space of AB is

$$\left\{ \begin{bmatrix} -2x \\ x \end{bmatrix} \mid x \in \mathbb{R} \right\}.\square$$

Problem 4.7.20: It once again is the case that BA is singular because B is singular. If you multiply a singular square matrix (on the left or the right) with any other square matrix (of the same size) then the result will again be singular. However, unlike problem 4.7.19, the null space is not the same as the null space of B. We will see that the null space of BA consists of those vectors $\vec{v}$ for which $A\vec{v}$ is in the null space of B. To see that this is the case, we see that if $A\vec{v}$ is in the null space of B, then

$$(BA)\vec{v} = B(A\vec{v}) = \vec{0},$$

and if $A\vec{v}$ is not in the nullspace of B, then

$$(BA)\vec{v} = B(A\vec{v}) \neq \vec{0}.$$

All of this abstract discussion was useful for understanding what is happening and how the null space of BA relates to the null space of B, but the easiest way to calculate the null space of BA (at this point in the course) is to calculate the matrix BA and to row reduce it as in problem 4.7.19. We see that

$$BA = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 3 & 1\cdot 2 + 2\cdot 4 \\ 2\cdot 1 + 4\cdot 3 & 2\cdot 2 + 4\cdot 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 14 & 20 \end{bmatrix}$$

Row reducing BA, we see that

$$\begin{bmatrix} 7 & 10 \\ 14 & 20 \end{bmatrix} \overset{R2-2R1}{\Longrightarrow} \begin{bmatrix} 7 & 10 \\ 0 & 0 \end{bmatrix}.$$

As discussed in problem 4.7.17, we see that the null space of BA is

$$\left\{ \begin{bmatrix} -\frac{10}{7}x \\ x \end{bmatrix} \mid x \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -10x \\ 7x \end{bmatrix} \mid x \in \mathbb{R} \right\}.\square$$

Bonus: The reader should check that for any vector $\vec{v}$ in the null space of BA, the vector $A\vec{v}$ is indeed in the null space of B.

Problem 4.7.21:
We can directly show that D is a singular matrix by producing a nonzero vector $\vec{v}$ for which $D\vec{v} = \vec{0}$. We see that for

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ we have } D\vec{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so D is a singular matrix.

Problem 4.7.24: We can directly show that E is a singular matrix by producing a nonzero vector $\vec{v}$ for which $E\vec{v} = \vec{0}$. We see that for

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ we have } E\vec{V} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so E is a singular matrix.

Problem 4.5.44: The matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

is an augmented matrix representing a system of equations. This system of equations can be expressed as a single vector equation as follows.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow$$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - x_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 + x_3 \\ 1 - 2x_3 \\ 1 \end{bmatrix}$$

so the vector form of the general solution (replacing x_3 with x) is

$$\left\{ \begin{bmatrix} -1+x \\ 1-2x \\ x \\ 1 \end{bmatrix} \mid x \in \mathbb{R} \right\} = \boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \text{x} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \text{ — x} \in \mathbb{R} \right\}} . \square$$