4.2.26) $x_1 - x_2 + x_3 = 4$
$2x_1 - 2x_2 + 3x_3 = 2$ $\rightarrow \begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & -2 & 3 & 2 \end{bmatrix}$

$R_2 - 2R_1$: $\begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix}$

$\downarrow$

$x_1 - x_2 + x_3 = 4$
$x_3 = -6$ $\rightarrow$ $x_1 - x_2 = 10 \rightarrow \boxed{x_1 = 10 + x_2}$
$x_3 = -6 \rightarrow \boxed{x_3 = -6}$

or so the solution set is $\boxed{\{(10+t, t, -6) : t \in \mathbb{R}\}.}$ $\square$

4.2.32) $x_1 + x_2 = 1$
$x_1 - x_2 = 3$ $\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$
$2x_1 + x_2 = 3$

$R_2 - R_1$, $R_3 - 2R_1$: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

$(-\frac{1}{2})R_2$, $(-1)R_3$: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$R_1 - R_2$, $R_3 - R_2$: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightarrow$ $\boxed{\begin{matrix} x_1 = 2 \\ x_2 = -1 \end{matrix}}$
$0 = 0$

4.2.35)

$$\begin{aligned} x_1 - x_2 - x_3 &= 1 \\ x_1 \quad + x_3 &= 2 \\ x_2 + 2x_3 &= 3 \end{aligned} \rightarrow \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$R_2 - R_1 : \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\rightarrow \quad x_1 - x_2 - x_3 = 1$$

$$\boxed{\begin{aligned} x_2 + 2x_3 &= 1 \quad \text{Inconsistent.} \\ x_2 + 2x_3 &= 3 \end{aligned}}$$

4.2.40)

$$\begin{aligned} x_1 + ax_2 &= 6 \\ ax_1 + 2ax_2 &= 4 \end{aligned} \rightarrow \begin{bmatrix} 1 & a & 6 \\ a & 2a & 4 \end{bmatrix}$$

$$R_2 - aR_1 = \begin{bmatrix} 1 & a & 6 \\ 0 & 2a-a^2 & 4-6a \end{bmatrix} \quad (\star)$$

If $2a-a^2 \neq 0$,

then

$$\left(\frac{1}{2a-a^2}\right) R_2 = \begin{bmatrix} 1 & a & 6 \\ 0 & 1 & \frac{4-6a}{2a-a^2} \end{bmatrix}$$

$$R_1 - aR_2 = \begin{bmatrix} 1 & 0 & 6 - \frac{4-6a}{2-a} \\ 0 & 1 & \frac{4-6a}{2a-a^2} \end{bmatrix} \rightarrow \begin{aligned} x_1 &= 6 - \frac{4-6a}{2-a}, \\ x_2 &= \frac{4-6a}{2a-a^2} \end{aligned}$$

So the only time we have no solutions is when $2a - a^2 = 0$ so that (☆) becomes

$\rightarrow \boxed{a = 0, 2}$

$$\left[\begin{array}{ccc} 1 & a & 6 \\ 0 & 0 & 4-6a \end{array}\right] \rightarrow \begin{array}{l} x_1 + a x_2 = 6 \\ 0 = 4 - 6a = 4 \text{ or } -8 \end{array}$$

4.2.42) $2\cos^2\alpha - \sin^2\beta = 1$ Let $x_1 = \cos^2\alpha$ and

$12\cos^2\alpha + 8\sin^2\beta = 13$, $x_2 = \sin^2\beta$.

Our new equations are

$$\begin{array}{l} 2x_1 - x_2 = 1 \\ 12x_1 + 8x_2 = 13 \end{array} \rightarrow \left[\begin{array}{ccc} 2 & -1 & 1 \\ 12 & 8 & 13 \end{array}\right]$$

$$R_2 - 6R_1 : \left[\begin{array}{ccc} 2 & -1 & 1 \\ 0 & 14 & 7 \end{array}\right]$$

$$\left(\tfrac{1}{14}\right) R_2 : \left[\begin{array}{ccc} 2 & -1 & 1 \\ 0 & 1 & \tfrac{1}{2} \end{array}\right]$$

$$R_1 + R_2 = \left[\begin{array}{ccc} 2 & 0 & 3/2 \\ 0 & 1 & \tfrac{1}{2} \end{array}\right]$$

$$\left(\tfrac{1}{2}\right) R_1 : \left[\begin{array}{ccc} 1 & 0 & 3/4 \\ 0 & 1 & 1/2 \end{array}\right] \rightarrow \begin{array}{l} x_1 = \tfrac{3}{4} \\ x_2 = \tfrac{1}{2} \end{array} \rightarrow \begin{array}{l} \cos^2\alpha = \tfrac{3}{4} \\ \sin^2\beta = \tfrac{1}{2} \end{array}$$

$$\rightarrow \begin{array}{l} \cos\alpha = \pm\tfrac{\sqrt{3}}{2} \\ \sin\beta = \pm\tfrac{1}{\sqrt{2}} \end{array} \rightarrow \boxed{(\alpha, \beta) \in \left\{ \left(\tfrac{\pi}{6}, \tfrac{\pi}{4}\right), \left(\tfrac{\pi}{6}, -\tfrac{\pi}{4}\right), \left(-\tfrac{\pi}{6}, \tfrac{\pi}{4}\right), \left(-\tfrac{\pi}{6}, -\tfrac{\pi}{4}\right) \right\}}$$

4.2.50) $(x,y)=(-1,6) \rightarrow \quad 6 = a\cdot(-1)^2 + b(-1)+c = a-b+c$

$(x,y)=(1,4) \rightarrow \quad 4 = a(1)^2 + b(1) + c = a+b+c$

$(x,y)=(2,9) \rightarrow \quad 9 = a(2)^2 + b(2)+c = 4a+2b+c.$

Therefore our system of equations is

$$\begin{aligned} a - b + c &= 6 \\ a + b + c &= 4 \\ 4a + 2b + c &= 9 \end{aligned} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 6 \\ 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 9 \end{bmatrix}$$

$R_2 - R_1,\ R_3 - 4R_1$: $$\begin{bmatrix} 1 & -1 & 1 & 6 \\ 0 & 2 & 0 & -2 \\ 0 & 6 & -3 & -15 \end{bmatrix}$$

$(\frac{1}{2})R_2,\ (\frac{1}{3})R_3$: $$\begin{bmatrix} 1 & -1 & 1 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & -1 & -5 \end{bmatrix}$$

$R_1 + R_2,\ R_3 - 2R_2$: $$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$(-1)R_3$: $$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$ $R_1 - R_3$: $$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

 $\rightarrow a = 2,\ b = -1,\ c = 3$, so $\boxed{y = 2x^2 - x + 3}$