Modified Problem 4.4.9: Find the general solution to the differential equation

$$y''' + y' = \sec(t). \tag{0.1}$$

Solution: We see that $1, \sin(t)$, and $\cos(t)$ are 3 linearly independent solutions to the homogeneous equation corresponding to equation (0.1). Letting $Y(t)$ denote the general solution to equation (0.1), we recall that

$$Y(t) = y_1(t)\int_0^t \frac{W_1(t)g(t)}{W(t)}dt + y_2(t)\int_0^t \frac{W_2(t)g(t)}{W(t)}dt + y_3(t)\int_0^t \frac{W_3(t)g(t)}{W(t)}dt \tag{0.2}$$

$$= 1\cdot\int_0^t \frac{W_1(t)\sec(t)}{W(t)}dt + \sin(t)\int_0^t \frac{W_2(t)\sec(t)}{W(t)}dt + \cos(t)\int_0^t \frac{W_3(t)\sec(t)}{W(t)}dt. \tag{0.3}$$

Noting that

$$W(t) = W(1,\sin(t),\cos(t)) = \begin{vmatrix} 1 & \sin(t) & \cos(t) \\ 0 & \cos(t) & -\sin(t) \\ 0 & -\sin(t) & -\cos(t) \end{vmatrix} \tag{0.4}$$

$$= 1\cdot\begin{vmatrix} \cos(t) & -\sin(t) \\ -\sin(t) & -\cos(t) \end{vmatrix} - 0\cdot\begin{vmatrix} \sin(t) & \cos(t) \\ -\sin(t) & -\cos(t) \end{vmatrix} + 0\cdot\begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} \tag{0.5}$$

$$= \begin{vmatrix} \cos(t) & -\sin(t) \\ -\sin(t) & -\cos(t) \end{vmatrix} = \cos(t)(-\cos(t)) - (-\sin(t))(-\sin(t)) = -1, \tag{0.6}$$

$$W_1(t) = W_1(1,\sin(t),\cos(t))(t) = \begin{vmatrix} 0 & \sin(t) & \cos(t) \\ 0 & \cos(t) & -\sin(t) \\ 1 & -\sin(t) & -\cos(t) \end{vmatrix} \tag{0.7}$$

$$= \begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} = \sin(t)(-\sin(t)) - \cos(t)\cos(t) = -1, \tag{0.8}$$

$$W_2(t) = W_2(1, \sin(t), \cos(t))(t) = \begin{vmatrix} 1 & 0 & \cos(t) \\ 0 & 0 & -\sin(t) \\ 0 & 1 & -\cos(t) \end{vmatrix} \tag{0.9}$$

$$= -\begin{vmatrix} 1 & \cos(t) \\ 0 & -\sin(t) \end{vmatrix} = -(1 \cdot (-\sin(t)) - 0 \cdot \cos(t)) = \sin(t), \text{ and} \tag{0.10}$$

$$W_3(t) = W_3(1, \sin(t), \cos(t)) = \begin{vmatrix} 1 & \sin(t) & 0 \\ 0 & \cos(t) & 0 \\ 0 & -\sin(t) & 1 \end{vmatrix} \tag{0.11}$$

$$= \begin{vmatrix} 1 & \sin(t) \\ 0 & \cos(t) \end{vmatrix} = 1 \cdot \cos(t) - 0 \cdot \sin(t) = \cos(t). \tag{0.12}$$

We now see that

$$Y(t) = \tag{0.13}$$
$$1 \cdot \int_0^t \frac{W_1(t)\sec(t)}{W(t)}dt + \sin(t)\int_0^t \frac{W_2(t)\sec(t)}{W(t)}dt + \cos(t)\int_0^t \frac{W_3(t)\sec(t)}{W(t)}dt$$

$$= \int_0^t \frac{-1 \cdot \sec(t)}{-1}dt + \sin(t)\int_0^t \frac{\sin(t)\sec(t)}{-1}dt + \cos(t)\int_0^t \frac{\cos(t)\sec(t)}{-1}dt \tag{0.14}$$

$$= \int_0^t \sec(t)dt - \sin(t)\int_0^t \tan(t)dt - \cos(t)\int_0^t 1dt \tag{0.15}$$

$$= \ln|\sec(t) + \tan(t)| + c_1 - \sin(t)(-\ln|\cos(t)| + c_2) - \cos(t)(t + c_3) \tag{0.16}$$

$$= \underbrace{(\ln|\sec(t) + \tan(t)| + \sin(t)\ln|\cos(t)| - t\cos(t))}_{y_p(t)} \tag{0.17}$$
$$+ \underbrace{(c_1 - c_2\sin(t) - c_3\cos(t))}_{y_c(t)}.$$

Problem 5.3.4: Let $y = \phi(x)$ be a solution to the initial value problem

$$y'' + x^2y' + \sin(x)y = 0; \quad y(0) = a_0, y'(0) = a_1. \tag{0.18}$$

Find $\phi''(0), \phi'''(0)$, and $\phi^{(4)}(0)$.

Solution: We proceed by trying to find a series solutions to equation (0.18) centered at $x = 0$. Letting

$$y(x) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots, \tag{0.19}$$

we see that $\phi^{(n)}(0) = n!a_n$, so we only need to determine a_2, a_3, and a_4. We also note that

$$y'(x) = \sum_{n=0}^{\infty} na_nx^{n-1} \overset{m=n-1}{=} \sum_{m=-1}^{\infty}(m+1)a_{m+1}x^m = \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots, \tag{0.20}$$

$$x^2y'(x) = \sum_{m=0}^{\infty}(m+1)a_{m+1}x^{m+2} \overset{k=m+2}{=} \sum_{k=2}^{\infty}(k-1)a_{k-1}x^k = a_1x^2 + 2a_2x^3 + 3a_3x^4 + \cdots, \tag{0.21}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} \overset{j=n-2}{=} \sum_{j=-2}^{\infty}(j+2)(j+1)a_{j+2}x^j = \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j = 2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\cdots, \text{ and} \tag{0.22}$$

$$\sin(x)y(x) = \left(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\right)(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots) \tag{0.23}$$

$$= x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) - \frac{x^3}{6}(a_0 + a_1x + \cdots) + \cdots \tag{0.24}$$

$$= a_0x + a_1x^2 + (a_2 - \frac{a_0}{6})x^3 + (a_3 - \frac{a_1}{6})x^4 + \cdots . \tag{0.25}$$

Combining the results of the previous calculations, we see that

$$\begin{aligned} 0 &= y'' + x^2y' + \sin(x)y \\ &= \left(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots\right) + \left(a_1x^2 + 2a_2x^3 + 3a_3x^4 + \cdots\right) \\ &\quad + \left(a_0x + a_1x^2 + (a_2 - \frac{a_0}{6})x^3 + (a_3 - \frac{a_1}{6})x^4 + \cdots\right) \end{aligned} \tag{0.26}$$

$$\begin{aligned} &= (2a_2) + (6a_3 + a_0)x + (12a_4 + 2a_1)x^2 + (20a_5 + 3a_2 - \frac{a_0}{6})x^3 \\ &\quad + (30a_6 + 4a_3 - \frac{a_1}{6})x^4 + \cdots \end{aligned} \tag{0.27}$$

$$\rightarrow \begin{array}{ccccccc} 2a_2 & & & & & = & 0 \\ a_0 & + & 6a_3 & & & = & 0 \\ 2a_1 & + & 12a_4 & & & = & 0 \\ -\frac{a_0}{6} & + & 3a_2 & + & 20a_5 & = & 0 \\ -\frac{a_1}{6} & + & 4a_3 & + & 30a_6 & = & 0 \end{array} \rightarrow (a_2, a_3, a_4) = (0, -\frac{a_0}{6}, -\frac{a_1}{6}) \tag{0.28}$$

$$\rightarrow \boxed{(\phi''(0), \phi'''(0), \phi^{(4)}(0)) = (0, -a_0, -4a_1)}. \tag{0.29}$$

Modified Problem 5.3.21: Solve the differential equation

$$y' + (x+1)y = x+1 \tag{0.30}$$

by finding a series solution and by using an integrating factor, then compare your answers.

Solution: We will first solve equation (0.30) by finding a series solution. We choose to find a series solution centered at $x = -1$ for convenience. Letting

$$y(x) = \sum_{n=0}^{\infty} a_n(x-(-1))^n = \sum_{n=0}^{\infty} a_n(x+1)^n = a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots \tag{0.31}$$

we see that

$$y'(x) = \sum_{n=0}^{\infty} na_n(x+1)^{n-1} \overset{m=n-1}{=} \sum_{m=0}^{\infty}(m+1)a_{m+1}(x+1)^m, \text{ and} \tag{0.32}$$

$$(x+1)y(x) = \sum_{n=0}^{\infty} a_n(x+1)^{n+1} \overset{k=n+1}{=} \sum_{k=1}^{\infty} a_{k-1}(x+1)^k. \tag{0.33}$$

Since

$$1\cdot(x+1) = y' + (x+1)y = \sum_{m=0}^{\infty}(m+1)a_{m+1}(x+1)^m + \sum_{k=1}^{\infty} a_{k-1}(x+1)^k \tag{0.34}$$

$$\overset{*}{=} a_1 + \sum_{n=1}^{\infty}\left((n+1)a_{n+1} + a_{n-1}\right)(x+1)^n, \tag{0.35}$$

we see that

$$\begin{array}{rcrcl} & & a_1 & = & 0 \\ 2a_2 & + & a_0 & = & 1 \\ (n+1)a_{n+1} & + & a_{n-1} & = & 0 \text{ for } n \geq 2 \end{array} \tag{0.36}$$

$$\rightarrow a_2 = \frac{1-a_0}{2}, a_{n+1} = -\frac{1}{n+1}a_{n-1} \text{ for } n \geq 2 \tag{0.37}$$

$$\rightarrow a_4 = -\frac{a_2}{4} = -\frac{1-a_0}{4\cdot 2}, a_6 = -\frac{a_4}{6} = \frac{1-a_0}{6\cdot 4\cdot 2}, a_8 = \cdots \tag{0.38}$$

$$\rightarrow a_n = \begin{cases} 0 & \text{if } n \text{ is odd.} \\ \frac{a_0-1}{(-2)^{\frac{n}{2}}(\frac{n}{2}!)} & \text{if } n \text{ is even and } n \geq 2 \end{cases} \tag{0.39}$$

It follows that the series solutions to equation (0.30) is

$$y(x) \overset{m=\frac{n}{2}}{=} a_0 + (a_0 - 1)\sum_{m=1}^{\infty}\frac{(x+1)^{2m}}{(-2)^m m!} = \boxed{1 + (a_0-1)\sum_{m=0}^{\infty}\frac{(x+1)^{2m}}{(-2)^m m!}}, \tag{0.40}$$

where a_0 can be determined by an initial condition if one is given.

We will now solve equation (0.30) by using an integrating factor. For convenience, we recall that equation (0.30) is

$$y' + (x+1)y = x+1. \tag{0.41}$$

Since the coefficient of y' is already 1, we see that the integrating factor $I(x)$ is given by

$$I(x) = e^{\int p(x)dx} = e^{\int (x+1)dx} \overset{*}{=} e^{\frac{(x+1)^2}{2}}. \tag{0.42}$$

Multiplying both sides of equation (0.41) by $I(x)$ yields

$$(x+1)e^{\frac{(x+1)^2}{2}} = e^{\frac{(x+1)^2}{2}}y' + (x+1)e^{\frac{(x+1)^2}{2}}y = (e^{\frac{(x+1)^2}{2}}y)' \tag{0.43}$$

$$e^{\frac{(x+1)^2}{2}}y = \int (x+1)e^{\frac{(x+1)^2}{2}}dx = e^{\frac{(x+1)^2}{2}} + c \tag{0.44}$$

$$\rightarrow y(x) = \boxed{1 + ce^{-\frac{(x+1)^2}{2}}}. \tag{0.45}$$

Recalling that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ we see that} \tag{0.46}$$

$$y(x) = 1 + ce^{-\frac{(x+1)^2}{2}} = 1 + c\sum_{n=0}^{\infty} \frac{(-\frac{(x+1)^2}{2})^n}{n!} = 1 + c\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{(-2)^n n!}. \tag{0.47}$$

By identifying m with n and identifying c with $a_0 - 1$, we see that both methods of solution yield the same answer.

Problem 5.3.7: Determine a lower bound for the radii of convergence r_1 and r_2 of the series solution to the differential equation

$$(1 + x^3)y'' + 4xy' + y = 0, \tag{0.48}$$

centered at $x_1 = 0$ and $x_2 = 2$. Then find the series solution to equation (0.48) centered at $x_2 = 2$.

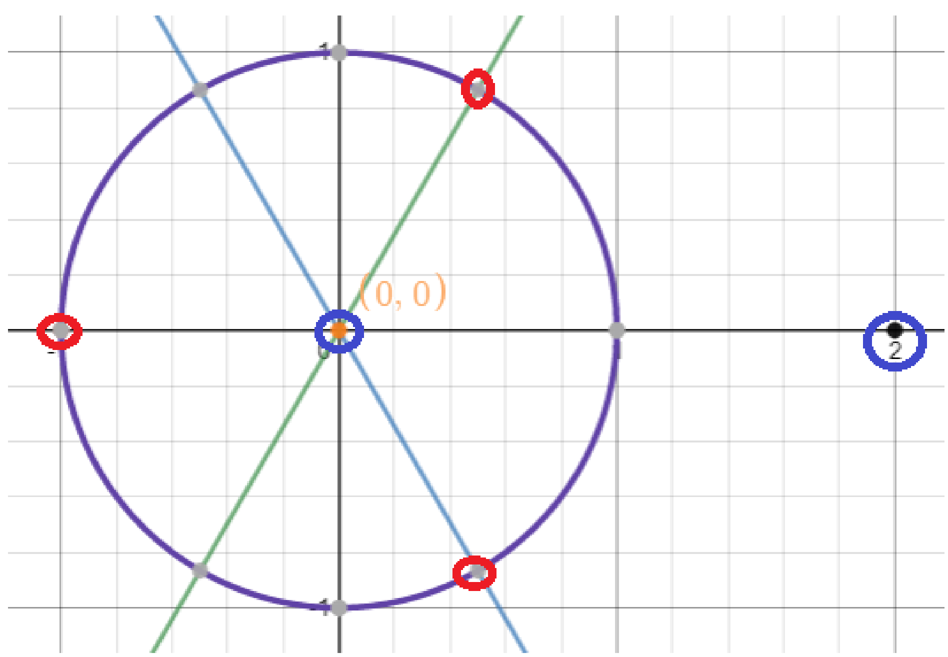
Solution: Firstly, we rewrite equation (0.48) in standard form to obtain

$$y'' + \frac{4x}{1+x^3}y' + \frac{1}{1+x^3}y = 0. \tag{0.49}$$

We see that as long as $1 + x^3 \neq 0$, then all coefficient functions of equation (0.49) are continuous. We see that for

$$x \in \{e^{\frac{\pi}{3}i}, e^{\pi i}, e^{\frac{5\pi}{3}i}\} = \{\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i\}, \text{ we have} \tag{0.50}$$

$$1 + x^3 = 1 + e^{\pi i} = 0. \tag{0.51}$$



We now see that all coefficient functions in equation (0.49) are continuous in a ball of radius 1 (in the complex plane) centered at the origin, so a series solution to equation (0.48) centered at $x = 0$ has a radius of convergence of at least 1. Similarly, we note that

$$|2 - (-1)| = 3, \tag{0.52}$$

$$|2 - (\frac{1}{2} + \frac{\sqrt{3}}{2}i)| = |\frac{3}{2} - \frac{\sqrt{3}}{2}i| = \sqrt{(\frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{3}, \text{ and} \tag{0.53}$$

$$|2 - (\frac{1}{2} - \frac{\sqrt{3}}{2}i)| = |\frac{3}{2} + \frac{\sqrt{3}}{2}| = \sqrt{(\frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{3}, \tag{0.54}$$

so the coefficient functions in equation (0.49) are continuous in a ball of radius $\sqrt{3}$ (in the complex plane) centered at 2, so the series solution to equation (0.48) centered at $x = 2$ has a radius of convergence of at least $\sqrt{3}$.

We will now begin finding the series solution to equation (0.48) centered at $x = 2$. Firstly, we note that we can rewrite equation (0.48) as follows.

$$0 = (1 + x^3)y'' + 4xy' + y = (1 + (x - 2 + 2)^3)y'' + 4(x - 2 + 2)y' + y \tag{0.55}$$

$$= (1 + (x - 2)^3 + 6(x - 2)^2 + 12(x - 2) + 8)y'' + 4(x - 2 + 2)y' + y \tag{0.56}$$

$$= (x - 2)^3y'' + 6(x - 2)^2y'' + 12(x - 2)y'' + 9y'' + 4(x - 2)y' + 2y' + y. \tag{0.57}$$

Since we are working with the series solution for $y = y(x)$ centered at $x = 2$, we have

$$y(x) = \sum_{n=0}^{\infty} a_n(x - 2)^n, \tag{0.58}$$

$$y'(x) = \sum_{n=0}^{\infty}(n + 1)a_{n+1}(x - 2)^n, \tag{0.59}$$

$$(x-2)y'(x) = \sum_{n=1}^{\infty} na_n(x-2)^n, \tag{0.60}$$

$$y''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-2)^n, \tag{0.61}$$

$$(x-2)y''(x) = \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-2)^n \tag{0.62}$$

$$(x-2)^2y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^n \tag{0.63}$$

$$(x-2)^3y''(x) = \sum_{n=3}^{\infty}(n-1)(n-2)a_{n-1}(x-2)^n, \text{ so} \tag{0.64}$$

$$0 = (x-2)^3y''+6(x-2)^2y''+12(x-2)y''+9y''+4(x-2)y'+2y'+y. \tag{0.65}$$

$$\begin{aligned} &= \sum_{n=3}^{\infty}(n-1)(n-2)a_{n-1}(x-2)^n + 6\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^n \\ &+ 12\sum_{n=1}^{\infty}(n+1)na_{n+1}(x-2)^n + 9\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-2)^n \\ &+ 4\sum_{n=1}^{\infty} na_n(x-2)^n + 2\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n + \sum_{n=0}^{\infty} a_n(x-2)^n \end{aligned} \tag{0.66}$$

$$\begin{aligned} &= (a_0+2a_1+18a_2)+(5a_1+28a_2+54a_3)(x-2)+(21a_2+82a_3+108a_4)(x-2)^2 \\ &+ \sum_{n=3}^{\infty}\Big((n-1)(n-2)a_{n-1} + 6n(n-1)a_n + 12(n+1)na_{n+1} \\ &+ 9(n+2)(n+1)a_{n+2} + 4na_n + 2(n+1)a_{n+1} + a_n\Big)(x-2)^n \end{aligned} \tag{0.67}$$

$$
\begin{aligned}
&(0.68)\\
&= (a_0+2a_1+18a_2)+(5a_1+28a_2+54a_3)(x-2)+(21a_2+82a_3+108a_4)(x-2)^2\\
&\quad + \sum_{n=3}^{\infty}\Big((n-1)(n-2)a_{n-1} + (6n^2-2n+1)a_n + (12n^2+14n+2)a_{n+1}\\
&\qquad + 9(n+2)(n+1)a_{n+2}\Big)(x-2)^n
\end{aligned}
$$

$$
(0.69)\quad \rightarrow \begin{array}{l}
a_0 = y(2)\\
a_1 = y'(2)\\
a_0 + 2a_1 + 18a_2 = 0\\
5a_1 + 28a_2 + 54a_3 = 0\\
21a_2 + 82a_3 + 108a_4 = 0\\
(n-1)(n-2)a_{n-1} + (6n^2-2n+1)a_n\\
+(12n^2+14n+2)a_{n+1} + 9(n+2)(n+1)a_{n+2} = 0 \text{ for } n \geq 3
\end{array}
$$

$$
(0.70)\quad \rightarrow \begin{array}{l}
a_0 = y(2)\\
a_1 = y'(2)\\
a_2 = -\frac{1}{9}a_1 - \frac{1}{18}a_0\\
a_3 = -\frac{28}{54}a_2 - \frac{5}{54}a_1\\
a_4 = -\frac{82}{108}a_3 - \frac{21}{108}a_2\\
a_{n+2} = \frac{1}{9(n+2)(n+1)}\Big((n-1)(n-2)a_{n-1}\\
+(6n^2-2n+1)a_n + (12n^2+14n+2)a_{n+1}\Big) \text{ for } n \geq 3
\end{array}.
$$

Once the recurrence in equations (0.70) is solved, our solution will be

$$
(0.71)\qquad y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n.
$$