**Problem 6.5.3:** Solve the initial value problem

$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t); \quad y(0) = 0, y'(0) = 0. \tag{0.1}$$

**Solution:** We recall that

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\}, \text{ and} \tag{0.2}$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) = s^2\mathcal{L}\{y(t)\}. \tag{0.3}$$

We now take the Laplace transform of both sides of equation (0.1) and see that

$$\mathcal{L}\{y''(t) + 3y'(t) + 2y(t)\} = \mathcal{L}\{\delta(t-5) + u_{10}(t)\} \tag{0.4}$$

$$\rightarrow \mathcal{L}\{y''(t)\} + 3\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t-5)\} + \mathcal{L}\{u_{10}(t) \cdot 1\} \tag{0.5}$$

$$\rightarrow s^2\mathcal{L}\{y(t)\} + 3s\mathcal{L}\{y(t)\} + 2\mathcal{L}\{y(t)\} = e^{-5s} + \frac{e^{-10s}}{s} \tag{0.6}$$

$$\rightarrow \mathcal{L}\{y(t)\} = \frac{e^{-5s} + \frac{e^{-10s}}{s}}{s^2 + 3s + 2} = \frac{se^{-5s} + e^{-10s}}{s(s+1)(s+2)}. \tag{0.7}$$

We will now use the method of partial fractions in order to break up the final expression in equation (0.7) into simpler components. We see that

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \tag{0.8}$$

$$= \frac{A(s+1)(s+2) + Bs(s+2) + Cs(s+1)}{s(s+1)(s+2)} \tag{0.9}$$

$$= \frac{(A+B+C)s^2 + (3A+2B+C)s + 2A}{s(s+1)(s+2)} \tag{0.10}$$

$$\begin{array}{ccccccc} A & + & B & + & C & = & 0 \\ \to 3A & + & 2B & + & C & = & 0 \\ 2A & & & & & = & 1 \end{array} \to A = \frac{1}{2} \tag{0.11}$$

$$\to \begin{array}{ccccc} B & + & C & = & -\frac{1}{2} \\ 2B & + & C & = & -\frac{3}{2} \end{array} \to B = (2B+C)-(B+C) = -1 \tag{0.12}$$

$$\to C = -\frac{1}{2} - B = \frac{1}{2} \to (A,B,C) = (\frac{1}{2}, -1, \frac{1}{2}). \tag{0.13}$$

$$\to \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}. \tag{0.14}$$

We also see that

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}. \tag{0.15}$$

It follows that

$$\mathcal{L}\{y(t)\} = \frac{se^{-5s}+e^{-10s}}{s(s+1)(s+2)} = \frac{e^{-5s}}{(s+1)(s+2)} + \frac{e^{-10s}}{s(s+1)(s+2)} \tag{0.16}$$

$$= \frac{e^{-5s}}{s+1} - \frac{e^{-5s}}{s+2} + \frac{\frac{1}{2}e^{-10s}}{s} + \frac{-e^{-10s}}{s+1} + \frac{\frac{1}{2}e^{-10s}}{s+2} \tag{0.17}$$

$$\begin{aligned} \to y(t) = \mathcal{L}^{-1}\{\frac{e^{-5s}}{s+1}\} - \mathcal{L}^{-1}\{\frac{e^{-5s}}{s+2}\} \\ + \frac{1}{2}\mathcal{L}^{-1}\{\frac{e^{-10s}}{s}\} - \mathcal{L}^{-1}\{\frac{e^{-10s}}{s+1}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{e^{-10s}}{s+2}\} \end{aligned} \tag{0.18}$$

$$\begin{aligned} = u_5(t)e^{-(t-5)}(t-5) - u_5(t)e^{-2(t-5)}(t-5) + \frac{1}{2}u_{10}(t)(t-10) \\ - u_{10}(t)e^{-(t-10)}(t-10) + \frac{1}{2}u_{10}(t)e^{-2(t-10)}(t-10) \end{aligned} \tag{0.19}$$

(0.20)

$$= \boxed{(e^{-(t-5)} - e^{-2(t-5)})u_5(t)(t-5) + (\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)})u_{10}(t)(t-10)}.$$

**Problem 6.6.18:** Solve the initial value problem

$$y'' + 3y' + 2y = \cos(\alpha t); \quad y(0) = 1, y'(0) = 0 \tag{0.21}$$

by using the Laplace transform and convolution integrals.

**Solution:** We recall that

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\} - 1, \text{ and} \tag{0.22}$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) = s^2\mathcal{L}\{y(t)\} - s. \tag{0.23}$$

We now take the Laplace transform of both sides of equation (0.21) and see that

$$\mathcal{L}\{y''(t) + 3y'(t) + 2y(t)\} = \mathcal{L}\{\cos(\alpha t)\} \tag{0.24}$$

$$\rightarrow \mathcal{L}\{y''(t)\} + 3\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{\cos(\alpha t)\} \tag{0.25}$$

$$\rightarrow (s^2\mathcal{L}\{y(t)\} - s) + 3(s\mathcal{L}\{y(t)\} - 1) + 2\mathcal{L}\{y(t)\} = \frac{s}{s^2 + \alpha^2} \tag{0.26}$$

$$\rightarrow (s^2 + 3s + 2)\mathcal{L}\{y(t)\} = \frac{s}{s^2 + \alpha^2} + s + 3 \tag{0.27}$$

$$\rightarrow \mathcal{L}\{y(t)\} = \frac{1}{(s+1)(s+2)}(\frac{s}{s^2 + \alpha^2} + s + 3) \tag{0.28}$$

$$= \frac{s}{(s^2 + \alpha^2)(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}. \tag{0.29}$$

As in the previous problem, we observe that

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \tag{0.30}$$

$$\to \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s+1} - \frac{s+3}{s+2} = (1+\frac{2}{s+1}) - (1+\frac{1}{s+2}) = \frac{2}{s+1} - \frac{1}{s+2}. \tag{0.31}$$

$$\to \mathcal{L}^{-1}\{\frac{s+3}{(s+1)(s+2)}\} = \mathcal{L}^{-1}\{\frac{2}{s+1} - \frac{1}{s+2}\} = 2\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{1}{s+2}\} = 2e^{-t} - e^{-2t}. \tag{0.32}$$

Another consequence of equation (0.30) is that

$$\mathcal{L}^{-1}\{\frac{1}{(s+1)(s+2)}\} = \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}\{\frac{1}{s+2}\} = e^{-t} - e^{-2t} \tag{0.33}$$

Since

$$\mathcal{L}^{-1}\{\frac{s}{s^2+\alpha^2}\} = \cos(\alpha t), \text{ we see that} \tag{0.34}$$

$$\mathcal{L}^{-1}\{\frac{s}{(s^2+\alpha^2)(s+1)(s+2)}\} = \mathcal{L}^{-1}\{\frac{s}{s^2+\alpha^2} \cdot \frac{1}{(s+1)(s+2)}\} \tag{0.35}$$

$$= \mathcal{L}^{-1}\{\frac{s}{s^2+\alpha^2}\} * \mathcal{L}^{-1}\{\frac{1}{(s+1)(s+2)}\} \tag{0.36}$$

$$\stackrel{*}{=} (\cos(\alpha t)) * (e^{-t} - e^{-2t}) \tag{0.37}$$

$$= \int_0^t \cos(\alpha u) \left(e^{t-u} - e^{2(t-u)}\right) du. \tag{0.38}$$

We now see that

$$y(t) = \mathcal{L}^{-1}\{\frac{s}{(s^2+\alpha^2)(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}\} \tag{0.39}$$

$$= \mathcal{L}^{-1}\{\frac{s}{(s^2+\alpha^2)(s+1)(s+2)}\} + \mathcal{L}^{-1}\{\frac{s+3}{(s+1)(s+2)}\} \tag{0.40}$$

$$= \boxed{\int_0^t \cos(\alpha u)\left(e^{t-u} - e^{2(t-u)}\right) du + 2e^{-t} - e^{-2t}}. \tag{0.41}$$

**Problem 4.1.17:** Show that $W(5, \sin^2(t), \cos(2t)) = 0$. Can this also be shown without directly computing the Wronskian?

**Solution:** We first proceed by direct calculation. Let $f(t) = 5$, $g(t) = \sin^2(t)$, and $h(t) = \cos(2t)$. We see that

$$f'(t) = f''(t) = 0, \tag{0.42}$$

$$g'(t) = 2\sin(t)\cos(t) = \sin(2t) \to g''(t) = 2\cos(2t), \text{ and} \tag{0.43}$$

$$h'(t) = -2\sin(2t) \to h''(t) = -4\cos(2t), \text{ so} \tag{0.44}$$

$$W(5, \sin^2(t), \cos(2t)) = W(f, g, h) = \begin{vmatrix} f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \\ f''(t) & g''(t) & h''(t) \end{vmatrix} \tag{0.45}$$

$$= \begin{vmatrix} 5 & \sin^2(t) & \cos(2t) \\ 0 & \sin(2t) & -2\sin(2t) \\ 0 & 2\cos(2t) & -4\cos(2t) \end{vmatrix} \tag{0.46}$$

$$= 5\begin{vmatrix} \sin(2t) & -2\sin(2t) \\ 2\cos(2t) & -4\cos(2t) \end{vmatrix} - 0 \cdot \begin{vmatrix} \sin^2(t) & \cos(2t) \\ 2\cos(2t) & -4\cos(2t) \end{vmatrix} + 0 \cdot \begin{vmatrix} \sin^2(t) & \cos(2t) \\ \sin(2t) & -2\sin(2t) \end{vmatrix} \tag{0.47}$$

$$= 5\begin{vmatrix} \sin(2t) & -2\sin(2t) \\ 2\cos(2t) & -4\cos(2t) \end{vmatrix} \tag{0.48}$$

$$= 5\left((\sin(2t)) \cdot (-4\cos(2t)) - (-2\sin(2t)) \cdot (2\cos(2t)\right) = 0. \tag{0.49}$$

Since $W(5, \sin^2(t), \cos(2t)) = 0$, we see that $5, \sin^2(t)$, and $\cos(2t)$ are linearly dependent. To find the linear dependence relation, we recall that $\cos(2t) = 1 - 2\sin^2(t)$, so

$$-\frac{1}{5} \cdot (5) + 2(\sin^2(t)) + (\cos(2t)) = -1 + 2\sin^2(t) + (1 - 2\sin^2(t)) = 0. \tag{0.50}$$

The linear dependence relation that is shown between $5, \sin^2(t)$, and $\cos(2t)$ in equation (0.50) is also sufficient for deducing that $W(5, \sin^2(t), \cos(2t)) = 0$.