Problem 3.1.23: Consider the differential equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0. \tag{0.1}$$

Find all values of α (if any) for which all solutions of equation (0.1) tend to zero as $t \to \infty$. Also find all values of α (if any) for which all nonzero solutions become unbounded as $t \to \infty$.

Solution: We see that the characteristic polynomial of equation (0.1) is

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1), \tag{0.2}$$

which has roots

$$r = \frac{2\alpha - 1 \pm \sqrt{(2\alpha - 1)^2 - 4\alpha(\alpha - 1)}}{2} = \frac{2\alpha - 1 \pm \sqrt{4}}{2} \tag{0.3}$$

$$= \alpha - \frac{5}{2}, \alpha + \frac{3}{2}. \tag{0.4}$$

Firstly, we note that the characteristic polynomial of equation (0.1) never has a double root, so the general solution is

$$y(t) = c_1 e^{(\alpha - \frac{5}{2})t} + c_2 e^{(\alpha + \frac{3}{2})t}. \tag{0.5}$$

The nonzero solutions of equation (0.1) will become unbounded as $t \to \infty$ if and only if $e^{(\alpha - \frac{5}{2})t}$ and $e^{(\alpha + \frac{3}{2})t}$ each become unbounded as $t \to \infty$. Recalling that $e^{\beta t}$ becomes unbounded as $t \to \infty$ if and only if $\beta > 0$, we see that the nonzero solutions of (0.1) become unbounded if and only if $\alpha - \frac{5}{2} > 0$, which occurs precisely when $\alpha > \frac{5}{2}$. Next, we see that the solutions of equations (0.1) tend to zero as $t \to \infty$ if and only if $\alpha - \frac{5}{2}$ and $\alpha + \frac{3}{2}$ are both negative, which occurs precisely when $\alpha < -\frac{3}{2}$.

Problem 3.1.24: Consider the differential equation

$$y'' + (3-\alpha)y' - 2(\alpha-1)y = 0. \tag{0.6}$$

Find all values of α (if any) for which all solutions of equation (0.6) tend to zero as $t \to \infty$. Also find all values of α (if any) for which all nonzero solutions become unbounded as $t \to \infty$.

Solution: We see that the characteristic polynomial of equation (0.6) is

$$r^2 + (3-\alpha)r - 2(\alpha-1), \tag{0.7}$$

which has roots

$$r = \frac{-(3-\alpha) \pm \sqrt{(3-\alpha)^2 - 4(-2(\alpha-1))}}{2} \tag{0.8}$$

$$= \frac{\alpha - 3 \pm \sqrt{\alpha^2 + 2\alpha + 1}}{2} = \frac{\alpha - 3 \pm (\alpha+1)}{2} = \alpha - 2, -2. \tag{0.9}$$

Since $r = -2$ is always a root of the characteristic polynomial, we see that e^{-2t} is always a solution to equation (0.6), so there are no values of α for which all nonzero solutions become unbounded as $t \to \infty$. Next, we note that the general solution to equation (0.6) is

$$y(t) = \begin{cases} c_1e^{-2t} + c_2e^{(\alpha-2)t} & \text{if } \alpha \neq 0 \\ c_1e^{-2t} + c_2te^{-2t} & \text{if } \alpha = 0 \end{cases}. \tag{0.10}$$

In order for all solutions to tend to zero as $t \to \infty$ we need $\alpha - 2$ to be negative, which occurs precisely when $\alpha < 2$.

Problem 3.2.29: Find the Wronskian of the differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0 \tag{0.11}$$

without solving the equation.

Solution: Firstly, we will divide both sides of equation (0.11) by t^2 to obtain

$$y'' - \left(\frac{t+2}{t}\right)y' + \left(\frac{t+2}{t^2}\right)y = 0. \tag{0.12}$$

Since equation (0.12) is a second order linear ordinary differential equation of the form

$$y'' + p(t)y' + q(t)y = g(t), \tag{0.13}$$

we see that a solution is gaurenteed to exist on $(-\infty, 0)$ or $(0, \infty)$. We also see that the Wronskian is

$$W(t) \stackrel{*}{=} e^{\int -p(t)dt} = e^{\int \frac{t+2}{t}dt} = e^{\int (1+\frac{2}{t})dt} \stackrel{*}{=} e^{t+2\ln(|t|)} = |t|^2e^t = \boxed{t^2e^t}. \tag{0.14}$$

We see that the Wronskian is never 0 on $(-\infty, 0)$ or $(0, \infty)$ so equation (0.12) has a unique solution for any initial values of the form $y(t_0) = c_1$ and $y'(t_0) = c_2$ with $t_0 \neq 0$ and $c_1, c_2 \in \mathbb{R}$.

Bonus Problem: Given that $y_1(t) = t$ is a solution to equation (0.11), use the Wronskian $W(t)$ to find another independent solution $y_2(t)$. (Compare with problem 3.4.26)

Solution: We see that

$$t^2e^t \stackrel{*}{=} W(t) \stackrel{*}{=} y_1y_2' - y_1'y_2 = ty_2' - y_2 \tag{0.15}$$

$$\to y_2' - \frac{1}{t}y_2 = te^t. \tag{0.16}$$

We can solve equation (0.16) by multiplying both sides by an integrating factor $I(t)$, which in this case is given by

$$I(t) = e^{\int -\frac{1}{t}dt} \overset{*}{=} e^{-\ln(|t|)} = \frac{1}{|t|}. \tag{0.17}$$

Since integrating factors are determined up to a constant, we may simply use $I(t) = \frac{1}{t}$ instead of $I(t) = \frac{1}{|t|}$. Multiplying both sides of (0.16) by $\frac{1}{t}$, we see that

$$e^t = \frac{1}{t}y_2' - \frac{1}{t^2}y_2 = (\frac{1}{t}y_2)' \tag{0.18}$$

$$\to \frac{1}{t}y_2 = \int e^t dt \overset{*}{=} e^t \tag{0.19}$$

$$\to \boxed{y_2(t) = te^t}. \tag{0.20}$$

Problem 3.3.19: Solve the initial value problem

$$y'' - 2y' + 5y = 0, \quad y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 2, \tag{0.21}$$

then sketch the graph of the solution and describe the behavior as $t \to \infty$.

Solution: We see that the characteristic polynomial of equation (0.21) is

$$r^2 - 2r + 5, \tag{0.22}$$

which has roots

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i. \tag{0.23}$$

It follows that the general solution to equation (0.21) is

$$y(t) = c_1' e^{(1+2i)t} + c_2' e^{(1-2i)t}, \tag{0.24}$$

which can also be more conveniently expressed as

$$y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t). \tag{0.25}$$

From the initial condition $y(\frac{\pi}{2}) = 0$ we see that

$$0 = y(\frac{\pi}{2}) = c_1 e^{\frac{\pi}{2}} \cos(\pi) + c_2 e^{\frac{\pi}{2}} \sin(\pi) = -c_1 e^{\frac{\pi}{2}} \to c_1 = 0. \tag{0.26}$$

From the initial condition $y'(\frac{\pi}{2}) = 2$ we see that

$$2 = y'(\frac{\pi}{2}) = \frac{d}{dt}(c_2 e^t \sin(2t))\Big|_{t=\frac{\pi}{2}} = (c_2 e^t \sin(2t) + 2c_2 e^t \cos(2t))\Big|_{t=\frac{\pi}{2}} \tag{0.27}$$

$$= c_2 e^{\frac{\pi}{2}} \sin(\pi) + 2c_2 e^{\frac{\pi}{2}} \cos(\pi) = -2c_2 e^{\frac{\pi}{2}} \to c_2 = -e^{-\frac{\pi}{2}} \tag{0.28}$$

$$\to \boxed{y(t) = -e^{-\frac{\pi}{2}} e^t \cos(2t) = -e^{t-\frac{\pi}{2}} \cos(2t)}. \tag{0.29}$$

We see from the graphs below that the solution $y(t)$ oscillates wildly as $t \to \infty$. Instead of converging to any particular value, the end behavior of $y(t)$ is unbounded and even oscillates between $-\infty$ and ∞.

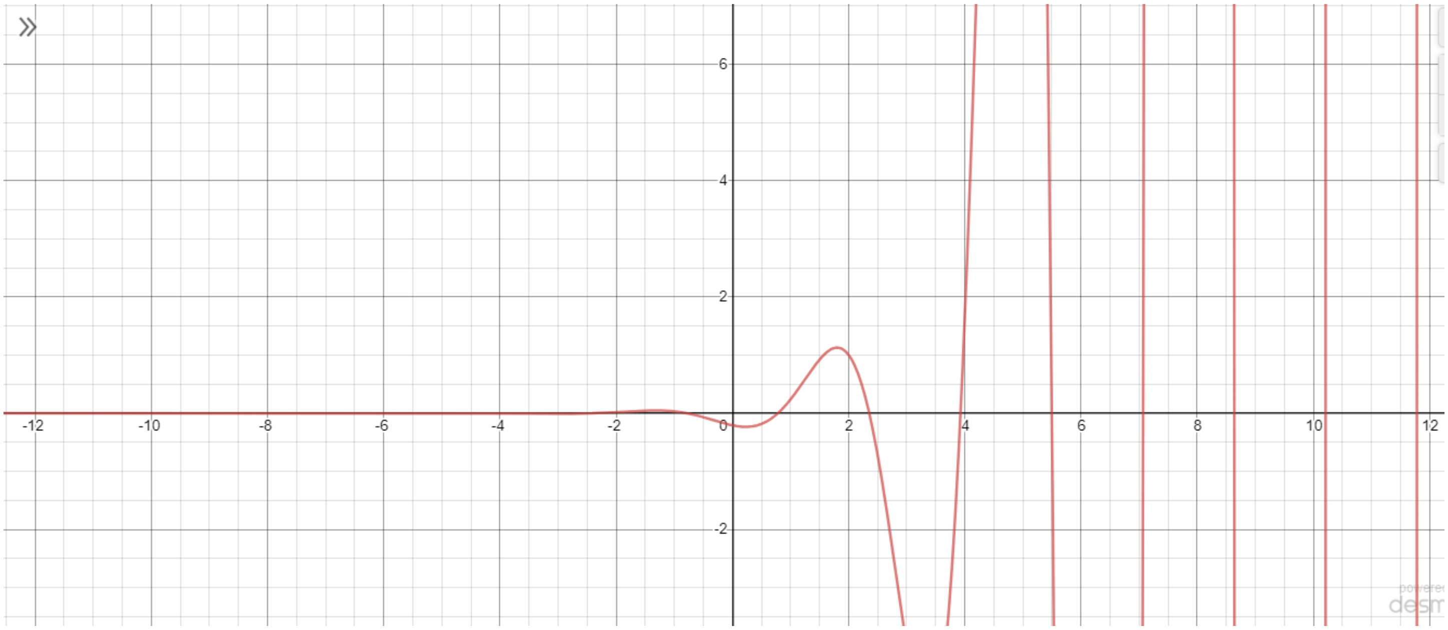

FIGURE 1. The graph of the solution $y(t)$ near the origin.

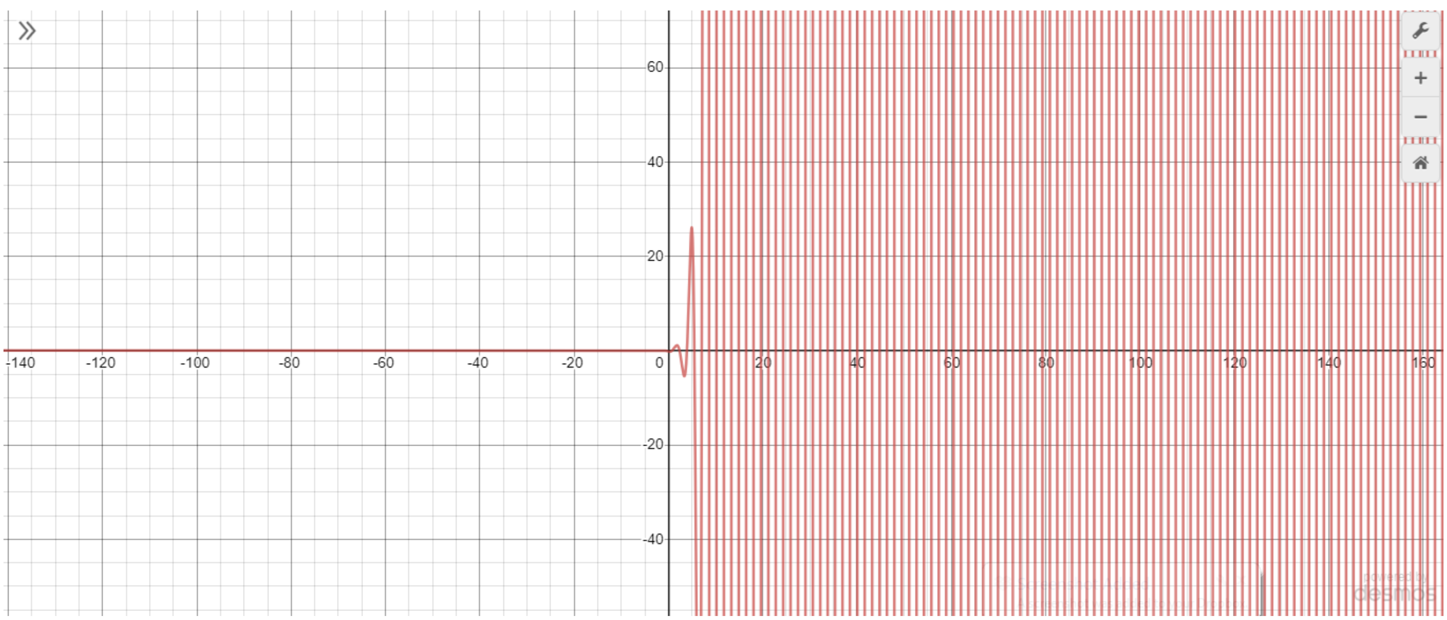

FIGURE 2. The graph of the solution $y(t)$ on a larger domain.

Problem 3.3.40: Solve the differential equation

$$t^2y'' - ty' + 5y = 0, \quad t > 0. \tag{0.30}$$

Solution: Since equation (0.30) is an Euler equation, we make the substitution $x = \ln(t)$ and $h(x) = y(e^x) = y(t)$. Since $t = e^x$, we may use the chain rule to see that

$$\frac{dh}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = e^x \frac{dy}{dt} = t\frac{dy}{dt}, \text{ and} \tag{0.31}$$

$$\frac{d^2h}{dx^2} = \frac{d}{dx}(\frac{dh}{dx}) = \frac{d}{dx}(e^x \frac{dy}{dt}) \tag{0.32}$$

$$= e^x \frac{dy}{dt} + e^x \left(\frac{d}{dx}\frac{dy}{dt}\right) = e^x \frac{dy}{dt} + e^x \left(\frac{d^2y}{dt^2} \cdot \frac{dt}{dx}\right) \tag{0.33}$$

$$= e^x \frac{dy}{dt} + e^x \left(e^x \frac{d^2y}{dt^2}\right) = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}. \tag{0.34}$$

We now see that substituting $x = \ln(t)$ into equation (0.30) yields

$$0 = t^2y'' - ty' + 5y = (t^2y'' + ty') - 2ty' + 5y = h'' - 2h' + 5h. \tag{0.35}$$

Since we now have t and x as independent variables, it is important to note that $h' = \frac{dh}{dx}$ and $y' = \frac{dy}{dt}$. This is not the most clear notation, so some people prefer to be more explicit and only write $\frac{dh}{dx}$ and $\frac{dy}{dt}$ without any use of $'$. Regardless of your preferred convention, be careful to avoid the errors that arise when you assume $y' = \frac{dy}{dx}$ and $h' = \frac{dh}{dt}$.

We see that the characteristic polynomial of equation (0.35) is

$$r^2 - 2r + 5, \tag{0.36}$$

and has roots

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i. \tag{0.37}$$

It follows that the general solution to equation (0.35) is

$$h(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x). \tag{0.38}$$

Finally, we see that

$$y(t) = h(x) = h(\ln(t)) = c_1 e^{\ln(t)} \cos(2\ln(t)) + c_2 e^{\ln(t)} \sin(2\ln(t)) \tag{0.39}$$

$$= \boxed{c_1 t \cos(2\ln(t)) + c_2 t \sin(2\ln(t))}. \tag{0.40}$$

Problem 3.4.20: Given $a \in \mathbb{R}$, solve the differential equation

$$y'' + 2ay' + a^2y = 0. \tag{0.41}$$

Hint: It helps to consider the Wronskian.

Solution: We see that the characteristic polynomial of equation (0.41) is

$$r^2 + 2ar + a^2 = (r+a)^2. \tag{0.42}$$

Since the characteristic polynomial has $r = -a$ as a repeated root, we see that one solution to equation (0.41) is $y_1(t) = e^{-at}$, but the second solution has yet to be found. To find the second solution, we will proceed as we did in the Bonus to problem 3.2.29. We see that the Wronskian is given by

$$W(t) = e^{\int -2adt} = e^{-2at}. \tag{0.43}$$

It follows that the second solution $y_2(t)$ satisfies the differential equation

$$e^{-2at} = W(t) = y_1y_2' - y_1'y_2 = e^{-at}y_2' + ae^{-at}y_2 \tag{0.44}$$

$$\to y_2' + ay_2 = e^{-at}. \tag{0.45}$$

We can solve equation (0.45) by multiplying both sides by an integrating factor $I(t)$. We see that

$$I(t) = e^{\int adt} = e^{at} \tag{0.46}$$

is a suitable choice of integrating factor. After multiplying both sides of equation (0.45) by e^{at}, we see that

$$1 = e^{at}y_2' + ae^{at}y_2 = (e^{at}y_2)' \tag{0.47}$$

$$\to e^{at}y_2 = t \to y_2 = te^{-at}. \tag{0.48}$$

Since $y_2(t)$ is indeed an independent solution to $y_1(t)$, we see that the general solution to equation (0.41) is

$$\boxed{y(t) = c_1 e^{-at} + c_2 t e^{-at}}. \tag{0.49}$$

It is clear that this solution is defined on all of $(-\infty, \infty)$. Furthermore, since the Wronskian $W(t)$ is never 0, we see that for any $t_0, b_1, b_2 \in \mathbb{R}$, there is a unique solution to equation (0.41) when we impose the initial conditions $y(t_0) = b_1$ and $y'(t_0) = b_2$.

Problem 3.4.26: Given that $y_1(t) = t$ is a solution to the differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0, \tag{0.50}$$

use the method of reduction of order to find a second solution. (Compare with problem 3.2.29)

Solution: Let $u(t)$ be such that $y_2(t) = u(t)y_1(t) = tu(t)$ is a second (independent) solution to equation (0.50). We see that

$$0 = t^2(tu(t))'' - t(t+2)(tu(t))' + (t+2)tu(t) \tag{0.51}$$

$$= t^2(tu''(t) + 2u'(t)) - t(t+2)(tu'(t) + u(t)) + (t+2)tu(t) \tag{0.52}$$

$$= t^3u''(t) + 2t^2u'(t) - t^3u'(t) - 2t^2u'(t) - t^2u(t) - 2tu(t) + t^2u(t) + 2tu(t) \tag{0.53}$$

$$= t^3u''(t) - t^3u'(t) \to 0 = u''(t) - u'(t) \tag{0.54}$$

$$\to u'(t) = u''(t) = \frac{du'(t)}{dt} \to dt = \frac{du'(t)}{u'(t)} \tag{0.55}$$

$$\to \int dt = \int \frac{du'(t)}{u'(t)} \tag{0.56}$$

$$\to t \overset{*}{=} \ln(u'(t)) \to u'(t) = e^t \tag{0.57}$$

$$\to u(t) = \int e^t dt \overset{*}{=} e^t. \tag{0.58}$$

It follows that a second solution to equation (0.50) is $y_2(t) = tu(t) = te^t$. After plugging te^t back into equation (0.50) to check our work, we see that $y_2(t) = te^t$ is indeed a second solution to equation (0.50) that is independent from $y_1(t) = t$.